MILNE'S INDUCTIVE ALGEBRA

Inductive Series.

THE INDUCTIVE ALGEBRA

EMBRACING

A COMPLETE COURSE

FOR

SCHOOLS AND ACADEMIES.

BY

WILLIAM J. MILNE, Ph. D., LL. D.,

Principal of State Normal School, Geneseo, N. Y.; Author of Milne's Inductive Arithmetics, etc.

NEW-YORK ∴ CINCINNATI ∴ CHICAGO
AMERICAN BOOK COMPANY

ECLECTIC EDUCATIONAL SERIES.

McGuffey's Revised Readers, Speller and Charts.
Milne's Inductive Arithmetics and Algebra.
Eclectic School Geometry.
Harvey's Revised Grammars.
Holbrook's Normal Grammars.
Ridpath's Inductive Grammar.
New Eclectic Geographies.
New Eclectic Penmanship.
Eclectic System of Drawing.
Forbriger's Drawing Tablets.
Eclectic Histories of the United States.
Thalheimer's Historical Series.
Ridpath's Histories of the United States.
Eclectic Complete Book-Keeping.
Murdoch's Analytic Elocution.
Kidd's New Elocution.
Etc., Etc., Etc.

Printed at
The Eclectic Press
Cincinnati. U. S. A.

PREFACE.

In this work on the Elements of Algebra, the author has followed the same plan that was adopted in his works on arithmetic. He has endeavored to present the subject in such a manner as to make it simple and attractive by rendering the transition from arithmetic to algebra easy and natural, while he has preserved all the valuable features which give discipline and skill in algebraic processes.

The method of teaching the subject, as given in this text-book, has been thoroughly tested in the class-room, and the results attained through its use have been more gratifying than could have been expected or hoped. The student is led, step by step, to a thorough and accurate comprehension of the principles of the science, and then they are fixed in the mind by abundant practice upon appropriate examples.

The order and treatment of the subjects will be found to be different from that given by most authors, yet it is confidently believed that the candid instructor will find the changes introduced to be of great assistance in interesting his students, and in inspiring them with a desire to investigate the beauties of this most attractive science.

The number of problems and examples given is unusually large, and the variety great. The definitions, principles, explanations, and demonstrations are brief, accurate, clear, and comprehensive, while they are free from the verbosity which commonly accompanies technical accuracy of statement. A cursory perusal of the work will disclose many new features, which the author feels sure will commend themselves to progressive and intelligent instructors.

With the hope that this book may prove a valuable aid alike to student and teacher in the investigation of THE SCIENCE OF ALGEBRA, the author presents his work to the public.

W. J. M.

STATE NORMAL SCHOOL,
GENESEO, N. Y., *January*, 1881.

CONTENTS.

ELEMENTS OF ALGEBRA.

ALGEBRAIC PROCESSES.

Article 1. Example 1. Two boys had together \$21. If the elder had twice as much as the younger, how much had each?

ARITHMETICAL SOLUTION.

A certain sum = what money the younger had.
2 times that sum = what money the elder had.
3 times that sum = what money both had.
Therefore, 3 times that sum = \$21.
The sum = \$7, what the younger had.
2 times \$7 = \$14, what the elder had.

The above solution may be abridged by using the letter s for the expressions, *a certain sum* and *that sum*. In Algebra it is common to use the letter x, or some other one of the last letters of the alphabet, for a number whose value is unknown, but is to be determined. Therefore, the following is the

ALGEBRAIC SOLUTION.

Let $x =$ money of the younger.
Then $2x =$ money of the elder.
And $3x =$ money of both.
Therefore, $3x = \$21$.
$x = \$7$, the money of the younger.
$2x = \$14$, the money of the elder.

DEFINITIONS.

2. An **Equation** is an expression of equality between two numbers or quantities.

Thus, $4 + 7 = 11$, and $2x = 16$, are equations.

3. A **Problem** is a question requiring solution.

4. A **Solution** of a problem is a process of finding the result sought.

5. A **Statement** of a problem is an equation which expresses the conditions of the problem.

Solve algebraically the following:

2. A man paid $30 for a coat and a vest. If the coat cost 4 times as much as the vest, what was the cost of each?

3. Two boys earned together $36. If James earned 3 times as much as Henry, how much did each earn?

4. A farmer picked 24 bushels of apples from two trees. If one tree bore twice as many bushels as the other, how many bushels did each bear?

5. A and B together furnish $800 capital, of which A furnishes 3 times as much as B. How much does each furnish?

6. A man had 450 sheep in three fields. In the second he had twice as many as in the first, and in the third 3 times as many as in the second. How many were there in each field?

7. Two boys together solved 350 problems, of which William solved 4 times as many as Charles. How many did each solve?

8. A certain number added to itself is equal to 260. What is the number?

9. A farmer sold a horse and a cow for $250, receiving 4 times as much for the horse as for the cow. How much did he receive for each?

10. A has 3 times as many sheep as B, and both have 420. How many has each?

11. A farm of 480 acres was divided between a brother and a sister, the brother having 3 times as many acres as the sister. How many acres had each?

12. The greater of two numbers is 5 times the less, and their sum is 540. What are the numbers?

13. A and B had a joint capital of $1750. A furnished 4 times as much as B. How much did each furnish?

14. A farmer raised 1320 bushels of grain. If he raised 5 times as much corn as wheat, how many bushels of each did he raise?

15. A farmer raised 1350 bushels of wheat, corn, and rye. If he raised twice as much corn as rye, and 3 times as much wheat as corn, how many bushels of each did he raise?

16. A, B, and C contributed $560 for the relief of the sick. A gave a certain sum, B gave twice as much as A, and C gave twice as much as B. How much did each give?

17. The number 169 can be divided into three integral parts such that the second part is 3 times the first, and the third 9 times the first. What are the parts?

18. The profits of a business for 3 years were $10890. The second year the gain was twice the gain of the first year, and the gain the third year was twice as much as that of both previous years. What was the gain the third year?

19. The expenses of a manufactory doubled each year for three years. The third year they were $13800. What were the expenses for each of the other years?

20. A lecturer received \$300 for 2 lectures. For the second lecture he received 3 times as much as he did for the first. How much did he receive for each?

21. A, B, and C own 10000 head of cattle. B owns 3 times as many as A, and C owns $\frac{1}{4}$ as many as are owned by A and B. How many does each own?

22. A number plus twice itself, plus 3 times itself, plus 4 times itself, equals 30. What is the number?

23. John has 5 times as many hens as ducks. He has in all 12 fowls. How many ducks has he?

24. A man has two daughters and one son. He wishes to divide \$6000 among them so as to give the elder daughter twice as much as the younger, and the son as much as both the daughters. How much must he give each?

25. Walter has 3 times as many slate-pencils as Albert has lead-pencils. The lead-pencils cost 3 cents apiece, and the slate-pencils 1 cent apiece, and together they cost 30 cents. How many slate-pencils has Walter?

26. Divide 36 into 4 parts so that the second shall be 8 times the first, the third shall be $\frac{1}{3}$ of the first and second, and the fourth shall be $\frac{1}{2}$ of the other three.

27. What number added to 5 times itself equals 90?

28. What number added to twice itself, and that sum added to 4 times the number, equals 28?

29. What number added to 7 times itself equals 104?

30. A and B enter into partnership to do business. A furnishes 4 times as much of the capital as B, and both together furnish \$15500. How much does each furnish?

31. A gentleman dying, bequeathed his property of \$14400 as follows: To his son 3 times as much as to his daughter, and to his widow twice as much as to both son and daughter. What was the share of each?

32. A farmer bought some grain for seed—in all, 32

bushels. He purchased 3 times as many bushels of oats as of barley, and as many bushels of wheat as of oats and barley. How many bushels of each kind did he purchase?

33. A merchant bought three pieces of cloth which together measured 144 yards. The second was 3 times as long as the first, and the third was 8 times as long as the first. What was the length of each piece?

34. A farmer had an orchard containing 560 trees. The number of peach trees was 3 times the number of cherry trees, and the number of apple trees 8 times the number of peach trees. How many were there of each?

35. James has 6 times as much money as John. He finds also that he has 30 cents more than John. How much has each?

36. A library contains 10000 volumes. The books of fiction are 9 times as many as the scientific works, the books of travel and biography each one-third as many as the books of fiction, and all the other works 4 times as many as the scientific works. How many books of fiction are there in the library?

37. Mary has 40 cents more than Sarah, and Mary's money is 5 times as much as Sarah's. How much money has each?

38. A farmer had 217 cattle in three fields. The first field contained twice as many as the third, and the second twice as many as the first. How many were there in each field?

39. The earnings of a manufactory doubled each year. If, at the end of four years, they amounted to $15000, what were the earnings the first year and the fourth year?

40. Three men engaged in business with a joint capital of $6000. A furnished three times as much as C, and B furnished $\frac{1}{2}$ as much as A and C. How much did each furnish?

INDUCTIVE EXERCISES.

6. 1. How many cubic feet are there in a block of marble containing 1 cubic yard?

2. What do the expressions *1 cubic yard* and *27 cubic feet* tell about the block of marble?

3. A cask was found to contain 35 gallons of water. What does the expression *35 gallons* tell about the water?

4. When it is said that a room contains 2000 cubic feet of space, what does the expression *2000 cubic feet* tell about the space?

5. When it is said that two places are 5 miles apart, what does the expression *5 miles* tell about the distance apart?

6. What may the *amount* or *extent* of any thing be called?

7. Name something that can be measured. Express some quantity of that thing.

8. In the expression *5 acres of land*, what expresses that which is measured? What expresses the quantity of land? What is that called which expresses the *quantity* or acres of land?

9. When any thing is measured, by what is the quantity expressed?

10. How will the price of any number of acres of land, at $10 per acre, compare with the number of acres? What expresses the quantity of land? What is that called which expresses the *quantity* or acres of land?

11. How do the expressions *5 acres* and *any number* of acres compare in definiteness?

12. In the problem, "How many dollars will 3 yards of cloth cost at $4 per yard," how many numbers are referred to? What numbers are given or known? What is the number *sought* or *unknown?*

DEFINITIONS AND SIGNS.

7. Quantity is the amount or extent of any thing.

Numbers are used to express quantity. In Algebra, however, the word *quantity* is frequently used for the word *number*.

8. Known Numbers, or **Quantities,** are such as have definite values, or those whose values are given, or to which any value can be assigned. They are represented by *figures* and the *first letters* of the alphabet.

Thus, 6, 8, 215, representing given numbers, and a, b, c, etc., representing any numbers, are known numbers, or quantities.

9. Unknown Numbers, or **Quantities,** are those whose values are to be found. They are represented by the *last letters* of the alphabet.

Thus, x, y, z, v, w, etc., are used to represent unknown numbers, or quantities.

10. Algebra is that branch of mathematics which treats of general numbers, or quantities, and the nature and use of equations.

The *Signs in Algebra* are, for the most part, the same as those used in Arithmetic.

11. The **Sign of Addition** is an upright cross: $+$. It is called *Plus*. Placed between quantities, it shows that they are to be added.

Thus, $a+b$ is read a plus b, and means that a and b are to be added.

12. The **Sign of Subtraction** is a short horizontal line: —. It is called *Minus*. Placed between two quanti-

ties it shows that the second is to be subtracted from the first.

Thus, $a - b$ is read a minus b, and means that b is to be subtracted from a.

13. The **Sign of Multiplication** is an oblique cross: $\times$. It is read *multiplied by* or *times*. Placed between two quantities, it shows that they are to be multiplied together.

Multiplication may also be indicated by a dot (.), or by writing the literal factors side by side.

Thus, $a \times b$, $a.b$, and ab, each shows that a is to be multiplied by b.

14. The **Sign of Division** is a short horizontal line between two dots: $\div$. It is read *divided by*. Placed between two quantities, it shows that the one at the left is to be divided by the one at the right.

Division may also be indicated by writing the dividend above the divisor, with a line between them.

Thus, $a \div b$ and $\frac{a}{b}$ each shows that a is to be divided by b.

15. The **Sign of Equality** is two short horizontal lines: $=$. It is read *equals*, or *is equal to*. When it is placed between two equal expressions an *Equation* is formed.

Thus, $a + b = 4$ is an equation.

16. The **Signs of Aggregation** are: The *Parenthesis*, ( ); the *Vinculum*, $\overline{\quad\quad}$; the *Bracket*, [ ]; and the *Brace*, { }. They show that the quantities included by them are to be subjected to the same process.

Thus, $(a+b)c$, $\overline{a+b} \times c$, $[a+b]c$, and $\{a+b\}c$, each shows that the sum of a and b is to be multiplied by c.

17. The **Sign of Involution** is a small figure or letter,

called an *Exponent*, written a little above and at the right of a quantity to indicate how many times the quantity is used as a factor.

Thus, a^5 shows that a is to be used as a factor 5 times, and is equal to $a \times a \times a \times a \times a$.

When no exponent is written, the exponent is 1.

Thus, a is regarded as a^1, b as b^1.

18. A **Power** of a quantity is the product arising from using the quantity a certain number of times as a factor.

Thus, 4 is the second power of 2; a^3 the third power of a.

19. Powers are *named* from the number of times the quantity is used as a factor.

Thus, a^5 is called the *fifth* power of a, or a fifth.

The *second* power of a quantity is also called the *square*, and the *third* power the *cube* of the quantity.

20. A **Root** of a quantity is one of the equal factors of the quantity.

Thus, 2 is a root of 4; a is a root of a^3.

21. Roots are *named* from the number of equal factors into which the quantity is separated.

Thus, one of *two* equal factors is the *second* root, one of *three* equal factors the *third* root, etc.

The *second* root of a quantity is also called the *square* root, and the *third* root the *cube* root of the quantity.

22. The **Sign of Evolution** is $\sqrt{}$, called the *Radical Sign*. When it is placed before a quantity it shows that a root of the quantity is required.

When no quantity or *Index* is written at the opening of the radical sign, the square root is indicated; if 3, as $\sqrt[3]{}$, the third root; if 4, as $\sqrt[4]{}$, the fourth root, etc.

Thus, $\sqrt[4]{a}$ is read the fourth root of a; $\sqrt[7]{b}$, the seventh root of b.

23. The **Ambiguous Sign** is $\pm$, a combination of the sign of Addition and the sign of Subtraction.

Thus, $a \pm b$ shows that b may be added to or subtracted from a.

24. A **Coëfficient** is a figure or letter placed before a quantity to show how many times the quantity is taken.

Thus, in the expression $7b$, 7 is the coëfficient of b, and it shows that $7b$ is equal to $b+b+b+b+b+b+b$.

In the expression $3ax$, 3 may be regarded as the coëfficient of ax, or $3a$ may be regarded as the coëfficient of x.

25. Coëfficients expressed by numbers are called **Numeral Coëfficients**; those expressed by letters, **Literal Coëfficients**; those expressed by figures and letters, **Mixed Coëfficients.**

When no coëfficient is expressed, the coëfficient is 1.

ALGEBRAIC EXPRESSIONS.

26. An **Algebraic Expression** is the expression of a quantity in algebraic language.

EXERCISES.

1. Interpret in ordinary language $a^2 + 3\sqrt{a^2 - x^2}$.

INTERPRETATION.—The algebraic expression interpreted or read is, the sum of a square and 3 times the square root of the remainder when x square is subtracted from a square. Or, the sum of a square and 3 times the square root of the quantity a square minus x square.

Copy and read the following expressions:

2. $a + b$.
3. $3b - a$.
4. $a^2 + b$.
5. $a - \sqrt{b}$.
6. $x^2 + b - c^2$.
7. $4(a + b) - c$.

8. $x^3 + \sqrt{x^5 - y}$.

9. $\sqrt{a + \sqrt{2b + c}}$.

10. $\dfrac{x + 4(x - 3y)}{2 - \sqrt{4x - z}}$.

11. $\sqrt{a + b} + x^2 - 4$

12. $\dfrac{\sqrt{a} + x + y^2}{\sqrt{a} - (x + y)}$.

13. $\dfrac{3x + y^2 - \sqrt{xy}}{4y^2 - z + 2xy^2}$.

When $a = 1$, $b = 2$, $c = 3$, $d = 4$, $e = 5$, find the numerical value of each of the following expressions by using the number for the letter which represents it:

Thus, $a + b + 3d - e = 1 + 2 + 12 - 5 = 10$.

1. $3a + b$.
2. $2c - b$.
3. $3d + a - b$.
4. $2c^2 - a - b$.
5. $d + c - 2a$.
6. $d - (a + b)$.
7. $a^2 + b^2 - d$.
8. $(a + b)d - c$
9. $(a + b)(d - c)$.
10. $(a^2 + b^2) \div (a + b)$.
11. $4(3a - b)$.
12. $7a(3d - 2a)$.
13. $abcd(a + b + c + d)$.
14. $(a + b + c)(a + b + c)$.
15. $(d + e - b) - (c - b)$.
16. $\sqrt{5e} + a + \dfrac{3b}{2}$.
17. $\left(\dfrac{a}{2} + \dfrac{c}{2}\right)d$.
18. $\dfrac{(a^2 + b^2)3b}{2e + a}$.
19. $a^2 + b^2 + c^2 + d^2 - e^2$.
20. $\sqrt{d} + (a + b)^2 - e$.
21. $(a + b)(b - a)4a$.
22. $a + 3\sqrt{2e + \sqrt{4e + 3d + 2b}}$.
23. $\left(\dfrac{ae}{a + c} + \dfrac{3d}{c}\right)^2$.
24. $\dfrac{3c(d^3 - c^3) - a}{2d + b}$.

DEFINITIONS.

27. The **Terms** of an algebraic expression are the parts connected by + or —.

Thus, in the expression $2a + 3x - 2cd$, there are three terms.

28. A **Positive Term** is one that has the sign + before it.

When the first term of an expression is positive, the sign $+$ is usually omitted.

Thus, in the expression $a + 3c - 2d + 5e$, the first, second, and fourth terms are *positive.*

29. A **Negative Term** is one that has the sign $-$ before it.

Thus, in the expression $3a - 2d - 3c + 2b - e$, the second, third, and fifth terms are *negative.*

30. **Similar Terms** are such as are formed of the same letters with the same exponents.

Thus, $3x^2$ and $12x^2$ are similar terms, as are also $2(x+y)^2$ and $4(x+y)^2$. ax^2 and bx^2 are similar terms when a and b are regarded as coëfficients.

31. **Dissimilar Terms** are such as contain different letters, or the same letters with different exponents.

Thus, $3xy$ and $2yz$ are dissimilar terms, as are also $3xy$ and $3xy^2$.

32. A **Monomial** is an algebraic expression consisting of one term.

Thus, xy, $3ab$, and $2y$ are monomials.

33. A **Polynomial** is an algebraic expression consisting of more than one term.

Thus, $x + y + z$ and $3a + 2b$ are polynomials.

34. A **Binomial** is a name applied to a polynomial of two terms.

Thus, $2a + 3b$ and $x - y$ are binomials.

35. A **Trinomial** is a name applied to a polynomial of three terms.

Thus, $x + y + z$ and $2a + 3b - 2c$ are trinomials.

ADDITION.

INDUCTIVE EXERCISES.

36. 1. How many apples are 5 apples, 3 apples, and 7 apples?

2. How many oranges are 5 oranges, 3 oranges, and 7 oranges?

3. How many things are 5 things, 3 things, and 7 things?

4. How many *a's* are $5a$, $3a$, and $7a$?

5. How many *b's* are $4b$, $3b$, $5b$, and $2b$?

6. How many *x's* are $3x$, $5x$, $9x$, $13x$, and $10x$?

7. How many *ab's* are $2ab$, $3ab$, $4ab$, $6ab$, and $9ab$?

8. How many a^2x's are $3a^2x$, $7a^2x$, $4a^2x$, and $2a^2x$?

9. How many a^3m^2's are $3a^3m^2$, $2a^3m^2$, $4a^3m^2$, and $9a^3m^2$?

10. James has no money, and owes one person 5 cents, another 3 cents, and another 2 cents. What is his financial condition?

11. If the sign $-$ is placed before each sum which he owes, what sign should be placed before the entire amount?

12. What financial condition is represented by -5 dollars, -7 dollars, -9 dollars, -3 dollars?

13. What sign will the sum of negative quantities have?

14. How many $-a$'s are $-9a$, $-3a$, $-7a$, $-8a$?

15. How many $-a^3x^4$'s are $-9a^3x^4$, $-7a^3x^4$, $-3a^3x^4$?

16. Asa owes one person 10 cents, another 12 cents, and another 15 cents. If James owes him 5 cents and Henry

owes him 9 cents, what is Asa's financial condition? What is the value of -10, -12, -15, 5, and 9?

17. How much is the debt in excess in the following: -8 dollars, -7 dollars, -9 dollars, 5 dollars, and 12 dollars?

18. Which is in excess, and how much, in the following: $3a$, $-5a$, $-2a$, $7a$, $-6a$, $9a$, $-2a$?

19. How many $(a+b)$'s are $2(a+b)$, and $3(a+b)$?

20. When no sign is prefixed to a number, or quantity, what sign is it assumed to have?

DEFINITIONS.

37. Addition is the process of uniting several quantities so as to express their value in the simplest form.

38. The **Sum** is the result obtained by adding.

39. PRINCIPLES.—1. *Only similar quantities can be united in one term.*

2. *Dissimilar quantities are added by writing them one after the other with their proper signs.*

CASE I.

40. To add similar monomials.

1. What is the sum of $3a$, a, $4a$, and $5a$?

PROCESS.

$$\begin{array}{r} 3a \\ a \\ 4a \\ 5a \\ \hline 13a \end{array}$$

EXPLANATION.—Since the quantities are *similar*—that is, have the same letter and same exponents—they are written in a column. The sum of $5a$, $4a$, a, and $3a$ is determined by adding the *coëfficients*, or numbers, which tell how many a's there are. Hence, the *sum* is $13a$.

Euclid

2. What is the value of $2a + 4a - 2a + 3a - a - 3a$?

PROCESS.

$2a$	$-2a$
$4a$	$-a$
$3a$	$-3a$
$9a$	$-6a$

$9a - 6a = 3a$

EXPLANATION.—Since the quantities are similar, they are written in columns.

The sum of the *positive* quantities is $9a$, and the sum of the *negative* quantities, or quantities to be subtracted, is $6a$.

$9a - 6a = 3a$. Hence, the value is $3a$.

Find the sum of each of the following:

(3.)	(4.)	(5.)	(6.)	(7.)
$4b$	$3ax$	$4x^2y$	$-4z^2y^2$	$-2cx^3$
b	$2ax$	$7x^2y$	$-3z^2y^2$	$-cx^3$
$7b$	ax	$3x^2y$	$-z^2y^2$	$-8cx^3$
$9b$	$4ax$	$2x^2y$	$-8z^2y^2$	$-cx^3$
$5b$	$9ax$	$9x^2y$	$-7z^2y^2$	$-cx^3$

8. Find the sum of ax, $3ax$, $7ax$, $9ax$, $8ax$, and $2ax$.

9. Find the sum of $7mn$, mn, $2mn$, $8mn$, $3mn$, and $5mn$.

10. Find the sum of $-3x^2y^2$, $-x^2y^2$, $-5x^2y^2$, $-7x^2y^2$, $-9x^2y^2$, and $-x^2y^2$.

11. Find the sum of $3x^3y^3$, $4x^3y^3$, $3x^3y^3$, x^3y^3, $7x^3y^3$, and x^3y^3.

12. Express $3a + 4a - 2a + 7a - 3a - 6a + a$ in the simplest form.

13. Express $9a^3x - 3a^3x + a^3x + 2a^3x - 7a^3x - a^3x$ in the simplest form.

14. Express $4\sqrt{xy} + 2\sqrt{xy} - 3\sqrt{xy} + \sqrt{xy} + 4\sqrt{xy} - 2\sqrt{xy}$ in the simplest form.

15. Express $3(xy)^3 + 4(xy)^3 - 3(xy)^3 - (xy)^3 - 7(xy)^3$ in the simplest form.

16. Express $2(x+y)^4 + 6(x+y)^4 - 7(x+y)^4 - 3(x+y)^4 - 4(x+y)^4 + 9(x+y)^4 - 9(x+y)^4 + 8(x+y)^4 - (x+y)^4$ in the simplest form.

CASE II.

41. To add when some terms are dissimilar.

1. Find the sum of $x+2y+z$, $x-y$, and $x+3y$.

PROCESS.

$$\begin{array}{l} x+2y+z \\ x-y \\ \underline{x+3y} \\ 3x+4y+z \end{array}$$

EXPLANATION.—For convenience in adding, similar terms are written in the same column. Since there are three different sets of similar quantities, their sum, or the simplest expression, is the sum of the different sets of quantities connected by their proper signs, for only similar quantities can be united in one term.

2. Express in its simplest form the following: $3x+2xy+z-3xy+2x-3z+4x-3xy-2xy+6z-7x+2w$.

PROCESS.

$$\begin{array}{rrrr} 3x & +2xy & +z & \\ 2x & -3xy & -3z & \\ 4x & -3xy & +6z & \\ -7x & -2xy & & +2w \\ \hline 2x & -6xy & +4z & +2w \end{array}$$

EXPLANATION.—The quantities are arranged so that similar terms are written in the same column. Beginning at either hand, each column is added separately, and the dissimilar terms of the result connected by their proper signs, for the dissimilar terms can not be united in one term.

RULE.—*Write similar terms in the same column. Add each column separately by finding the difference of the sums of the positive and negative terms. Connect the results with their proper signs.*

EXAMPLES.

(3.)

$$\begin{array}{rrr} 3a & +2b & \\ -2a & +3b & -c \\ 2a & & +2c \\ & 3b & -7c \\ 3a & -4b & \\ \hline \end{array}$$

(4.)

$$\begin{array}{rrr} 5x & +3xy & \\ 2x & -7xy & \\ -3x & -6xy & \\ & 4xy & -3z \\ 3x & & +4z \\ \hline \end{array}$$

(5.)

$$\begin{array}{rrr} 3x & +4z & -xz \\ 2x & -4z & \\ & 3z & -4xz \\ 3x & +6z & -4xz \\ & & 7xz \\ \hline \end{array}$$

Express in their simplest form the following:

6. $3x + 2y - 3z - 2y + 3z - 6x + 4y + 3z + 3x + 3z - 6y.$
7. $4xy + z - y + 3z - y - 3xy + xy - y + z + 4x - 3y + z.$
8. $3ac + 4ay + 2ac - 3ay + 2ay + 2ac - 3ac + ay.$
9. $9b + 2cd - 3e - 3cd + 9b + 3cd - 6e - 2b - 4e + 3cd.$
10. $3x^2y + 3xy - 3z + 6xy - 6x^2y + 2z - 3xy + 6z - 4z.$
11. $a + 6b + 3c - 4a + 3c + 3a - 6b + d + 2c - 3a + 7d.$
12. $x^2y + y + w - 3y + 2w + 2x^2y + z - 3x^2y - 3y + 2w.$
13. $9a^2b^2 - 3c^3y^3 + 2d^2 - 4c^3y^3 + 4a^2b^2 - 3d^2 + 2d^2 - 3a^2b^2.$

14. Add $3ab + 3\sqrt{xy} + 4$, $4\sqrt{xy} - 2ab + 7$, $7ab + 3 + 2\sqrt{xy}$, $2\sqrt{xy} + 4 - 4ab$, and $3ab - 2\sqrt{xy} + 7$.

15. Add $3x^3 - 4x^2 - x + 7$, $2x^3 - x^2 + 3x - 10$, $2x^2 - 7x^3 - 2x + 4$, $3x^3 - 2x^2 + 12 - 3x$, $11x^3 + 5x^2 + 6x - 7$.

16. Add $\frac{3}{4}ax^2 + \frac{2}{3}a^2 + 2x^3y + b^3$, $3ax^2 + \frac{1}{4}x^3y + 3a^2 - 2b^3$, $2ax^2 + 3x^3y - a^2 - \frac{2}{3}b^3$, and $\frac{1}{16}ax^2 + \frac{1}{8}x^3y + 3a^2 - \frac{2}{5}b^3$.

17. Add $ac^2 + ab^2 + \frac{1}{4}a^3 - a^2b + \frac{3}{4}abc + \frac{1}{3}a^2c$, $a^2b + b^3 + ab^2 + bc^2 + 2abc + \frac{1}{2}b^2c$, and $a^2c - ac^2 + b^2c - bc^2 + c^3 + abc$.

18. Add $3(x + y)$, $4(x + y)$, $9(x + y)$, $-10(x + y)$, $3(x + y)$, $-5(x + y)$, $7(x + y)$, and $-3(x + y)$.

19. Add $5(a - b)^2 + 3(x - y)^2$, $4(x - y)^2 - 2(a - b)^2$, $7(a - b)^2 - 3(x - y)^2$, and $5(x - y)^2 - 3(a - b)^2$.

20. What is the sum of $4x^3 + ax^3 - bx^3 + 2x^3$?

PROCESS.

$$\begin{array}{r} +4x^3 \\ +ax^3 \\ -bx^3 \\ +2x^3 \\ \hline (6 + a - b)x^3 \end{array}$$

EXPLANATION.—Since the dissimilar terms have a common factor, x^3, 4, a, $-b$, and 2 may be regarded as the coëfficients of x^3, and their sum, which is $6 + a - b$, will be the coëfficient of x^3 in the sum.

Therefore, the sum is $(6 + a - b)x^3$.

21. What is the sum of $2ax - 3bx + 4cx + 3dx$?

22. What is the sum of $2ax^2 - 4bx^2 + 3cx^2 + 4x^2$?

23. Add $2(a+b)$, $3a(a+b)$, $4(a+b)$, and $2a(a+b)$.

24. Add $5(a+3)$, $2(a+3)$, $3a(a+3)$, and $2b(a+3)$.

25. Add $3a\sqrt{x+y}$, $2\sqrt{x+y}$, $2a\sqrt{x+y}$, and $3\sqrt{x+y}$.

EQUATIONS AND PROBLEMS.

42. Simplify the following and find the value of x:

1. $3x + 4x + 2x - 3x - 2x + 4x = 16$.

SOLUTION.

$$3x + 4x + 2x - 3x - 2x + 4x = 16$$

Uniting terms, $\quad 8x = 16$

Whence, $\quad x = 2$

2. $5x + 2x - 3x + 4x - 6x + 7x = 18$.
3. $5x + 6x - 9x - 3x + 2x + 4x = 20$.
4. $3x - 2x + 5x + 7x + 4x - 3x = 26 + 2$.
5. $3x - 4x + 2x + 6x - 4x + x = 15 + 3 - 2$.
6. $x + 4x + 6x - 3x + 7x - 9x = 21 + 7 - 4$.
7. $9x - 2x - 3x + 7x - 5x + 4x = 35 + 9 - 4$.
8. $8x - 4x + 7x + 3x - 6x - 4x = 37 - 3 + 2$.
9. $11x - 3x + 7x - 4x + 6x - 3x = 23 + 7 - 2$.
10. $10x - 4x + 2x + 7x - 6x + 2x = 35 + 6 + 3$.

Solve the following problems:

11. James solved twice as many problems as Henry, and Henry solved 3 times as many as Harvey. If they all solved 70 problems, how many did each solve?

12. A had twice as much money as B, and B had twice as much as C. If they all had $140, how much had each?

13. William had twice as many marbles as Henry, and Henry had 3 times as many as Samuel. How many had each, if they all had 50 marbles?

14. A merchant owes B a certain sum of money, and C twice as much. Various persons owe him in all 10 times as much as he owes B. After paying all his debts he will have $1400 left. How much does he owe B and C?

15. After taking 5 times a number from 13 times a number and adding to the remainder 8 times the number, the result was 5 more than 155. What was the number?

16. A circulating library contained 10 times as many books of reference and 3 times as many historical books as works of fiction. The works of reference exceeded the works of fiction and history by 12000 volumes. How many volumes were there of each?

17. A merchant failed in business, owing A 10 times as much as B, C three times as much as B, and D twice the difference of his indebtedness to B and C. The entire debt to these persons was $36000. How much did he owe each?

18. At a local election there were three candidates for an office who polled the following vote respectively: A received twice as many as B, and B $1\frac{1}{2}$ times as many as C. The vote for all lacked 3 votes of being 1125. How large a vote did each receive?

19. A man earned daily for 5 days 3 times as much as he paid for his board, after which he was obliged to be idle 4 days. Upon counting his money after paying for his board he found that he had 2 ten-dollar bills and 4 dollars. How much did he pay for his board, and what were his wages?

SUBTRACTION.

INDUCTIVE EXERCISES.

43. 1. What is the difference between 5 dollars and 3 dollars?

2. What is the difference between 7 miles and 9 miles?

3. What is the difference between $9m$ and $3m$?

4. What is the remainder when $8a$ is taken from $12a$?

5. What is the remainder when $3a^2xy$ is taken from $8a^2xy$?

6. What is the remainder when $8x^3y^2z^4$ is taken from $15x^3y^2z^4$? What is the sum of $-8x^3y^2z^4$ and $15x^3y^2z^4$?

7. What is the remainder when $9y^4x^2$ is taken from $18y^4x^2$? What is the sum of $-9y^4x^2$ and $18y^4x^2$?

8. What is left when $3a^2b$ is taken from $12a^2b$? What is the sum of $12a^2b$ and $-3a^2b$?

9. What is left when $5pq^2$ is taken from $13pq^2$? What is the sum of $13pq^2$ and $-5pq^2$?

10. What is left when $3(x+y)$ is taken from $8(x+y)$? What is the sum of $8(x+y)$ and $-3(x+y)$?

11. How much less than zero is -2? -4? -9?

12. When 8 is subtracted from 0, how is the result expressed? If 2 should be subtracted from that result, what would be the result? How many are -8 less 2? $-8a$ less $2a$? What is the sum of $-8a$ and $-2a$?

13. What is the result when $7x^2y$ is taken from $-3x^2y$? What is the sum of $-3x^2y$ and $-7x^2y$?

14. Instead of subtracting a *positive* quantity, what may be done to secure the same result?

15. What is the result when $7-3$ is subtracted from 13? 7 from 13? $8a-5a$ from $11a$? $8a$ from $11a$? $9x^2-2x^2$ from $12x^2$? $9x^2$ from $12x^2$?

16. How does the result, when $7-3$ is subtracted from 13, compare with the result when 7 is subtracted from 13? How does the result, when $8a-5a$ is subtracted from $11a$, compare with the result, when $8a$ is subtracted from $11a$?

17. What is the result when $7-3$ is subtracted from 8? What is the sum of $8-7+3$?

18. What is the result when $6x^2-3x^2$ is subtracted from $8x^2$? What is the sum of $8x^2-6x^2+3x^2$?

19. What is the remainder when $6xy-4xy$ is subtracted from $9xy$? What is the sum of $9xy-6xy+4xy$?

20. Instead of subtracting a *negative* quantity, what may be done to secure the same result?

DEFINITIONS.

44. Subtraction is the process of finding the difference between two quantities; or,

The process of finding a quantity which, added to one given quantity, will produce another.

45. The **Minuend** is the quantity from which another is to be subtracted.

46. The **Subtrahend** is the quantity to be subtracted.

47. The **Difference,** or **Remainder,** is the result obtained by subtracting.

48. PRINCIPLES.—1. *The difference between similar quantities only can be expressed in one term.*

2. *Subtracting a positive quantity is the same as adding a numerically equal negative quantity.*

3. *Subtracting a negative quantity is the same as adding a numerically equal positive quantity.*

CASE I.

49. To subtract when the terms are positive.

1. From $9a$ subtract $3a$.

PROCESS.

$$\begin{array}{r} 9a \\ -\underline{3a} \\ 6a \end{array}$$

EXPLANATION.—When 3 times any number is subtracted from 9 times that number, the remainder is 6 times the number; therefore, when $3a$ is subtracted from $9a$, the remainder is $6a$. Or, since subtracting a positive number or quantity is the same as adding an equal negative quantity (Prin. 2), $3a$ may be subtracted from $9a$ by changing the sign of $3a$ and adding the quantities. Therefore, to subtract $3a$ from $9a$, we find the sum of $9a$ and $-3a$, which is $6a$.

2. From $13a$ take $15a$.

PROCESS.

$$\begin{array}{r} 13a \\ -\underline{15a} \\ -2a \end{array}$$

EXPLANATION.—After subtracting from $13a$ as much as we can of $15a$, there will be $2a$ yet to be subtracted, or the result will be $-2a$. Or, since subtracting a positive quantity is the same as adding an equal negative quantity (Prin. 2), $15a$ may be subtracted from $13a$ by finding the sum of $13a$ and $-15a$, which is $-2a$. Therefore, when $15a$ is taken from $13a$, the result is $-2a$.

	(3.)	(4.)	(5.)	(6.)	(7.)	(8.)
From	$15a$	$13xy$	$15x^3y^2$	$19xyz$	$3x^2y^3z$	$10a^2b^3c$
Take	$6a$	$8xy$	$17x^3y^2$	$22xyz$	$15x^2y^3z$	$13a^2b^3c$

Copy and subtract the following:

9. $8x + 2y$ from $12x + 6y$.
10. $9a + 3b$ from $10a + 2b$.
11. $7xy + 2z$ from $5xy + 4z$.
12. $3x^2y^2 + 6z$ from $5x^2y^2 + 3z$.
13. $8xy^3z + 3xy$ from $6xy^3z + 2xy$.
14. $4p^2qs + 3pq^2s$ from $5p^2qs + 6pq^2s$.
15. $5m^2nx + 3mnx$ from $7m^2nx + 2mnx$.
16. $5x^2y + 2y^2$ from $9x^2y + 7y^2$.
17. $3xy^2 + 4z$ from $xy^2 + z$.
18. $5p^2q^2 + 5pq$ from $p^2q^2 + 4pq$.
19. $x^2y^2z^2 + 4y^2$ from $15x^2y^2z^2 + 2y^2$.
20. $8yz^4 + y^4z$ from $3yz^4 + 3y^4z$.
21. $3p^2q^2 + 4qs$ from $9p^2q^2 + 2qs$.
22. $10xyz^3 + 4xyz$ from $xyz^3 + xyz$.
23. $8a^2xy + 5ax^2$ from $4a^2xy + 7ax^2$.
24. $9r^2s^2z + 8rsz^2$ from $10r^2s^2z + 4rsz^2$.

CASE II.

50. To subtract when some terms are negative.

1. From $6a$ subtract $-3a$.

PROCESS.

$$\begin{array}{r} 6a \\ -3a \\ \hline 9a \end{array}$$

EXPLANATION.—If 0 were subtracted from $6a$, the remainder would be $6a$; therefore, when $-3a$, which is $3a$ less than 0, is subtracted from $6a$, the difference, or remainder, is $9a$. Or, since subtracting a negative quantity is the same as adding an equal positive quantity (Prin. 3), we may subtract by changing the sign of $-3a$ and adding the quantities, obtaining for a result $9a$.

2. From $6a - 2b$ subtract $3a - 4b$.

PROCESS.

$$\begin{array}{r} 6a - 2b \\ 3a - 4b \\ \underline{- \quad +} \\ 3a + 2b \end{array}$$

EXPLANATION.—The subtrahend is written under the minuend, so that similar terms stand in the same column. Since the subtrahend is composed of two terms, each term must be subtracted separately. Subtracting $3a$ from $6a - 2b$ leaves $3a - 2b$, or the result may be obtained by adding $-3a$ to $6a - 2b$. But since the subtrahend was $4b$ less than $3a$, to obtain the true remainder, $4b$ must be added to $3a - 2b$, which gives $3a + 2b$. Therefore, the subtraction may be performed by changing the sign of each term of the subtrahend and adding the quantities.

RULE.—*Write similar terms in the same column. Change the sign of each term of the subtrahend from + to —, or from — to +, or conceive it to be changed, and proceed as in Addition.*

PROOF.—*Add together the remainder and the subtrahend. If the result is equal to the minuend, the work is correct.*

	(3.)	(4.)	(5.)	(6.)	(7.)
From	$4a^3x$	$3x^3y^3$	$2x + y$	$6y - 2z$	$7ax - 4by$
Take	$-2a^3x$	$-5x^3y^3$	$-2x - 2y$	$3y + 4z$	$3ax - 9by$

	(8.)	(9.)	(10.)
From	$3a + 2b - 3c$	$4x + 3y - 3z$	$4xy + 3z + x^2$
Take	$2a - 4b + 5c$	$2x - 4y - 5z$	$2xy - 3z + 4x^2 - y$

11. From $a + b + c$ subtract $a + 2b - c$.
12. From $3x + 2y - 3z$ subtract $2x - 3y + 4z$.
13. From $6a^2 + 2b^2 + 3c^2$ subtract $3a^2 - 3b^2 - 2c^2$.
14. From $3a^3 - 2c^3 - 4d^3$ subtract $4c^3 - 3a^3 + 2d^3$.
15. From $8x^4 - 3y^2 + 2z^3$ subtract $4y^2 - 3x^4 + 2z^3$.
16. From $9p^2 + 4q^2 + r^3$ subtract $3r^3 - 4p^2 - 2q^2$.
17. From $ax + 2ay + z$ subtract $2ax - 2ay + z$.

18. From $2xy + 5yz + 3xz$ subtract $2xy - 3yz - 4xz$.

19. From $8x^3y^2 + 16xy^3 + 10xy$ subtract $14x^3y^2 - 8xy^3 - 4xy$.

20. From $5x^3y^3 + 10x^4y - 6yz^3$ subtract $10x^4y - 4x^3y^3 + 5yz^3$.

21. From $3x^2 + 2xy + z^2 + w$ subtract $2x^2 - 3xy - 4z^2$.

22. From $15x^3 + 10y^3 + 8z^3 - r^3$ subtract $5y^3 + 4z^3 + 6r^3$.

23. From $4xy^2 + 3x^5y + 4x - 3$ subtract $4xy^2 - 3x - 7$.

24. From $4bx^3 + 3ay^2 + 4 - by$ subtract $by - 5 - bx^3$.

25. From $3x^2y^4 + 3xy - 5x$ subtract $2x^2y^4 - 2xy + 4x - 5$.

26. From $4x^5y^2 - 3xy^5 - 7z^4$ subtract $2x^5y^2 + 6xy^5 + 2z^4 + 9$.

27. From $7ar^2 - 4bs^3 + 3rs$ subtract $3ar^2 + p + 2bs^3 + 7$.

28. From $15x^5 - 24x^3y^3 - 16y^4$ subtract $15x^3y^3 + 4z - 5y^4 + z^5$.

29. From $3x^m - 4x^ny^m + 4y^m$ subtract $4x^m + 2x^ny^m - 4x^{2m}$.

30. From $3x^{2m} - 2x^{3n}y^m - y^{m-1}$ subtract $3y^{m-1} + 2x^{3n}y^m - 4x^{2m}$.

31. From $3\sqrt{xy} + 2z - \sqrt[3]{y^2}$ subtract $2\sqrt{xy} - 3z - 2\sqrt[3]{y^2}$.

32. From $4(a+b)^2 - 3a + 4c$ subtract $a - 2(a+b)^2 - 2c$.

33. From $5\sqrt{a+b^2} - 3\sqrt[3]{x+y}$ subtract $6\sqrt[3]{x+y} - 7\sqrt{x+y}$.

34. From $5\sqrt{a+b^2} - 3\sqrt[3]{c+d}$ subtract $4\sqrt{a+b^2} + 2\sqrt[3]{c+d}$.

35. From $ax + by$ subtract $cx - dy$.

PROCESS.

$$\begin{array}{c} ax + by \\ cx - dy \\ \hline (a-c)x + (b+d)y \end{array}$$

EXPLANATION.—Since the terms, though dissimilar, have a common factor, a and c may be regarded as the coëfficients of x, and b and d as the coëfficients of y, and the difference indicated by placing the difference between the coëfficients in parentheses.

36. From $ay + 2x$ subtract $cy - dx$.

37. From $(a + b)x + (c + d)y$ subtract $2(a + b)x - 3(c + d)y$.

38. From $(a - b)x + (a + b)y$ subtract $(a - c)x - (b - c)y$.

39. From $ax + by - z$ subtract $bx - ay - cz$.

40. From $5ay + 2cz - 6x$ subtract $cy - az - dx$.

41. From $ax^2 + 2cy + 3x^3y$ subtract $2bx^2 - 3ay - cx^3y$.

THE PARENTHESIS.

51. The subtrahend is sometimes placed in a parenthesis, or between brackets, and written after the minuend with the sign $-$ between them.

Thus, when $b + c - d$ is subtracted from $a + b$, the result is sometimes indicated as follows: $a + b - (b + c - d)$.

1. What must be done to the subtrahend to perform the operation indicated in the expression $a - (b + c)$?

2. What must be done to the subtrahend to perform the operation indicated in the expression $x + y - (c + d - e)$?

3. What must be done to the subtrahend to perform the operation indicated in the expression $3x - y - (-2x - y)$?

4. What change must be made in the signs of the terms of the subtrahend in each of the last three examples when the parenthesis is removed?

5. When a quantity inclosed in a parenthesis, in brackets, or similar signs, is preceded by the sign $-$, what change must be made in the signs of the terms when the parenthesis or other similar sign is removed?

52. PRINCIPLES.—1. *A parenthesis, preceded by the minus sign, may be removed from a quantity by changing the signs of all the terms of the quantity.*

2. *A parenthesis, preceded by the minus sign, may be used to inclose a quantity by changing the signs of all the terms of the quantity.*

When quantities are inclosed in a parenthesis preceded by the *plus* sign, the parenthesis may be removed without any change of signs, and consequently any number of terms may be inclosed in a parenthesis with the *plus* sign without any change of signs.

The student should remember that in expressions like $-(x^2-y+z)$ the sign of x^2 is plus, and the expression is the same as if written $-(+x^2-y+z)$.

Simplify the following:

1. $a-(a+b)$.
2. $x-(x-y)$.
3. $a+b-(-a)$.
4. $a-(-a-b)$.
5. $a-(a-b)$.
6. $y-(-x-y)$.
7. $4a-(2a+y)$.
8. $3x+2y-(2x-2y)$.
9. $5x-3y-(-2x+4y)$.
10. $7x+3z-(x+y+z)$.
11. $2x-3y^2z-(x+z^2-3y^2z)$.
12. $3xy+2x^3y-(4xy-x^3y+x^2)$.
13. $3x^2+2y^2-(-4x^2-2y^2-z^2)$.
14. $3ab^2-2ac^2-(-3ab^2-6ac^2)$.

When the expression contains more than one parenthesis, they may be removed *in succession* by beginning with the outside one or inside one.

Thus,
$$\begin{aligned} &a+b-(c-a+[d+b]-c+2b-d)= \\ &a+b-c+a-[d+b]+c-2b+d = \\ &a+b-c+a-d-b+c-2b+d = \\ &2a-2b. \end{aligned}$$

Simplify the following:

15. $a - b - c - (d + 2a + [3b - 2c + d] - 4a - 2b)$.

16. $x^2 + 2y - (x^3 + [2y + 3x^2 - 4x^3] - 6y + 3x^2) + 4x^3$.

17. $-(x^2y + 2y - 3) - (x^2y - [6y + 7 - 3x^2y] + 9)$.

18. $ab + bc - (3ab + [3bc + 2bd - 3ab] + 2bd) - 6c$.

19. $-\{3ax - [2xy + 3z] + z - (4xy + [3ax + 6z] + 3z)\}$.

TRANSPOSITION IN EQUATIONS.

53. 1. If a certain number plus 10 equals 25, what is the number?

2. If a certain number minus 5 equals 10, what is the number?

3. A number $- 6 = 13$. What is the number?

4. A number $+ 6 = 13$. What is the number?

5. If $x + 2 = 10$, what is the value of x?

6. If $x - 2 = 10$, what is the value of x?

7. If $x - 5 = 20$, what is the value of x?

8. If $x + 5 = 20$, what is the value of x?

9. In the equation $x - 5 = 20$, what is done with the 5 in obtaining the value of x? In the equation $x = 20 + 5$, how does the sign of the 5 compare with its sign in the previous equation?

10. In the equation $x + 5 = 20$, what is done with the 5 in obtaining the value of x? In the equation $x = 20 - 5$, how does the sign of the 5 compare with its sign in the previous equation?

11. In changing the 5's from one side, or member, of the equation to the other, what change was made in the sign?

12. When a number, or quantity, is changed from one member of an equation to the other, what change must be made in its sign?

13. If 5 is added to one member of the equation $2+3=5$, what must be done to the other member so as to preserve the equality?

14. If 5 is subtracted from one member of the equation $2+3=5$, what must be done to the other member so as to preserve the equality?

15. If one member of the equation $2+3=5$ is multiplied by 5, what must be done to the other member to preserve the equality?

16. If one member of the equation $2+3=5$ is divided by 5, what must be done to the other member to preserve the equality?

17. If one member of the equation $7+9=16$ is raised to the second power, or if the second root of one member is found, what must be done to the other member to preserve the equality?

18. What, then, may be done to the members of an equation without destroying the equality?

DEFINITIONS.

54. Members of an Equation are the parts on each side of the sign of equality.

55. The **First Member** of an equation is the part on the left of the sign of equality.

56. The **Second Member** of an equation is the part on the right of the sign of equality.

57. Transposition is the process of changing a quantity from one member of an equation to the other.

58. An **Axiom** is a truth that does not need demonstration.

59. Axioms.—1. *The same quantity may be added to both members of an equation without destroying the equality.*

2. *The same quantity may be subtracted from both members of an equation without destroying the equality.*

3. *Both members of an equation may be multiplied by the same quantity without destroying the equality.*

4. *Both members of an equation may be divided by the same quantity without destroying the equality.*

5. *Both members of an equation may be raised to the same power, or the same root of both members may be extracted, without destroying the equality.*

60. Principle.—*A quantity may be transposed from one member of an equation to the other by changing its sign from + to —, or from — to +.*

EQUATIONS AND PROBLEMS.

61. 1. $2x - 3 = x + 6$. Find the value of x.

PROCESS.

$$\begin{array}{rl} 2x - 3 &= x + 6 \\ +3 &= +3 \\ \hline 2x &= x + 9 \\ x &= x \\ \hline x &= 9. \end{array}$$

OR,

$$\begin{aligned} 2x - 3 &= x + 6 \\ 2x - x &= 6 + 3 \\ x &= 9 \end{aligned}$$

Explanation.—Since the known and unknown quantities are found in both members of the equation, in order to find the value of x, the known quantities must be collected in one member and the unknown in the other.

Since -3 is found in the first member, it may be caused to disappear by adding 3 to both members (Axiom 1), which gives the equation $2x = x + 9$.

Since x is found in the second member, it may be caused to disappear by subtracting x from both members (Axiom 2), which gives, as a resulting equation, $x = 9$.

Or, since a quantity may be changed from one member of an equation to the other by changing its sign (Prin.), -3 may be

transposed to the second member by changing it to $+3$, and x may be transposed to the first member by changing it to $-x$. Therefore, the resulting equation will be $2x - x = 6 + 3$. By uniting the terms the result is $x = 9$.

The result may be *verified* by substituting the *value* of x for x in the original equation. If both members are then identical, the value of the unknown quantity is correct. Thus, substituting 9 for x in the original equation, it becomes $18 - 3 = 9 + 6$, or $15 = 15$. Therefore, the value of x is 9.

RULE.—*Transpose the terms so that the unknown quantities stand in the first member of the equation, and the known quantities in the second.*

Unite similar terms, and divide the equation by the coëfficient of the unknown quantity.

VERIFICATION.—*Substitute the value of the unknown quantity in the original equation. If both members are then identical in value, the value of the unknown quantity found is correct.*

1. The same quantity, with the same sign upon opposite sides of an equation, may be cancelled from both.

2. The value of an equation will not be changed if the signs of all the terms are changed at the same time.

Transpose, and find the value of x in the following:

2. $x + 3 = 7$.
3. $2x - 4 = 12$.
4. $2x - 10 = 14$.
5. $3x + 7 = 28$.
6. $3x - 5 = 25$.
7. $7x - 3 = 25$.
8. $9x + 6 = 24$.
9. $8x - 13 = 27$.
10. $7x + 5 = 26$.
11. $10x - 5 = 35$.
12. $12x + 6 = 30$.
13. $13x - 4 = 35$.
14. $2x + 2 = 6 + x$.
15. $3x - 4 = 6 + x$.
16. $3x + 5 = 11 - x$.
17. $4x + 2 = 3x + 8$.
18. $4x - 11 = 9 - x$.
19. $4x + 3 = 3x + 10$.

20. $7x - 5 = 19 + 4x.$
21. $9x - 3 = 30 - 2x.$
22. $35 + 2x = 5x + 2.$
23. $25 - 10 = 24 + 3x - 15.$
24. $10x - 3x = 13 + 4x + 38.$
25. $3x - 6 = x + 14 - 4.$

Solve the following problems:

26. What number increased by 9 is equal to 34?

PROCESS.

Let x represent the number.

Then, by the conditions of the problem, $x + 9 = 34$

Transposing, $x = 34 - 9$

Uniting similar terms, $x = 25$

27. What number diminished by 15 equals 31?

28. What number increased by 9 equals 27?

29. What number diminished by 10 equals 33?

30. What number added to twice itself gives a sum equal to 45?

31. What number added to three times itself gives a sum equal to 72?

32. What number is there whose double exceeds the number by 10?

33. What number is there such that, if 10 be added to it, twice the sum will be 44?

34. Twice a certain number increased by 4 is equal to the number increased by 15. What is the number?

35. Three times a certain number diminished by 5 is equal to the number plus 21. What is the number?

36. A man walked 71 miles in three days, walking 3 miles more the second day than the first, and 5 miles more the third day than the second. How far did he travel each day?

PROCESS.

Let x = the number of miles he traveled the 1st day.
Then, $x+3=$ the number of miles he traveled the 2d day.
And, $x+8=$ the number of miles he traveled the 3d day.
Therefore, $x+x+3+x+8=71$
Transposing, $x+x+x=71-3-8$
Uniting similar terms, $3x=60.$
Whence, $x=20$, the number of miles he traveled the 1st day
$x+3=23$, the number of miles he traveled the 2d day
$x+8=28$, the number of miles he traveled the 3d day

37. Three boys had together 85 cents. James had 10 cents more than John, and Henry had 5 cents more than James. How much had each?

38. A farmer remembered that he had 395 sheep distributed in three fields, so that there were 20 more in the second than in the first, and 25 more in the third than in the second, but he could not tell how many there were in each field. Find the number in each field.

39. A drover being asked if he had 100 head of cattle, replied that if he had twice as many as he then had and 4 more he would have 100. How many had he?

40. A gentleman left his estate, amounting to $6900, to be divided among his four sons, so that each should have $150 more than his next younger brother. How much was the share of each?

41. The expenses of a manufacturer for 4 years were $9500. An examination showed an annual increase of $250. What were his yearly expenses?

MULTIPLICATION.

INDUCTIVE EXERCISES.

62. 1. If a man earns 2 dollars per day, how much will he earn in 5 days?

2. If a man walks 4 miles per hour, how far will he walk in 3 hours?

3. How many m's are 3 times $4m$? 2 times $4m$? 5 times $6m$?

4. If a boy can gather 3 quarts of chestnuts per hour, how many quarts can he gather in 4 hours? How many q's are 4 times $3q$?

5. A vessel sails 6 miles north per hour, indicated by $+6$. How far will she sail in 3 hours? What sign should be placed before the product to indicate the direction sailed?

6. How many are 3 times $+6$? 3 times $+6a$? 2 times $+5b$? 3 times $+7x$? 4 times $+3a^2$?

7. If a vessel sails 5 miles south per hour, indicated by -5, how far will she sail in 4 hours? What sign should be placed before the product to indicate the direction sailed?

8. How many $-m$'s are 4 times $-5m$? 3 times $-6m$? How many are 5 times $-4b$? 6 times $-3x$?

9. When a quantity is written without any sign before it, what sign is it understood to have?

10. How many are 6 times $+4m$, or $+6$ times $+4m$? 5 times $+3x$, or $+5$ times $+3x$? 8 times $-2y$, or $+8$ times $-2y$? 5 times $-3x$, or $+5$ times $-3x$?

11. When a positive quantity is multiplied by a positive quantity, what is the sign of the product?

12. When a negative quantity is multiplied by a positive quantity, what is the sign of the product?

13. Multiply $+3$ by 2 or $+2$; $+3a$ by 6 or $+6$; $+5x$ by $+3$; -3 by 5 or $+5$; $-3a$ by 2 or $+2$; $-6x$ by $+4$; $-5xy$ by $+8$.

14. How does the product of 4×5 compare with the product of 5×4? 6×7 with 7×6? $+4a\times +5$ with $+5\times +4a$? $-4\times +3$ with $+3\times -4$? What, then, is the product of $-4\times +3a$? Of $+3a\times -4$? Of $-5x\times +7$? Of $+7\times -5x$?

15. When a positive quantity is multiplied by a negative quantity, what is the sign of the product?

16. What is the product of -3×6?

17. Since -3×6 is -18, if -3 is multiplied by $6-2$, how many times -3 must be subtracted from -18 to obtain the true result?

18. If the subtraction is indicated, what are the signs of the remainder when -6 is subtracted from -18?

19. What is the product of -5×4?

20. Since -5×4 is -20, if -5 is multiplied by $4-3$, how many times -5 must be subtracted from -20? If the subtraction is indicated, what are the signs of the remainder when -15 is subtracted from -20?

21. Since, in the results just obtained, -3 multiplied by -2 gives $+6$ and -5×-3 gives $+15$, what may be inferred as to the sign of the product when a negative quantity is multiplied by a negative quantity?

22. What is an exponent? What does it show? In the expression 5^3, what does the 3 show? In the expression a^5, what does the 5 show? In a^6, what does the 6 show?

23. When a^3 is multiplied by a^2, how many times is a used as a factor? How many times is a used as a factor when a^2 is multiplied by a^5?

24. How, then, may the number of times a quantity is used as a factor in multiplication, be determined from the exponents of the quantity in the expressions which are multiplied?

25. How is the exponent of a quantity in the product determined?

DEFINITIONS.

63. **Multiplication** is the process of repeating one quantity as many times as there are units in another.

64. The **Multiplicand** is the quantity to be repeated or multiplied.

65. The **Multiplier** is the quantity showing how many times the multiplicand is to be repeated.

66. The **Product** is the result obtained by multiplying.

67. The multiplicand and multiplier are called the *factors* of the *product*.

68. The **Signs of Multiplication.** (See Art. 13.)

69. PRINCIPLES.—1. *Either factor may be used as multiplier or multiplicand when both are abstract.*

2. *The sign of any term of the product is + when its factors have* LIKE *signs, and — when they have* UNLIKE *signs.*

3. *The coëfficient of a quantity in the product is equal to the product of the coëfficients of its factors.*

4. *The exponent of a quantity in the product is equal to the sum of its exponents in the factors.*

70. The principle relating to the signs of the terms of the product is illustrated as follows:

$$\begin{aligned} +a \text{ multiplied by } +b &= +ab \\ -a \text{ multiplied by } +b &= -ab \\ +a \text{ multiplied by } -b &= -ab \\ -a \text{ multiplied by } -b &= +ab \end{aligned}$$

CASE I.

71. To multiply when the multiplier is a monomial.

1. What is the product of $3a^2x$ multiplied by $2a^3x^2y$?

PROCESS.

$$\begin{array}{l} 3a^2x \\ \underline{2a^3x^2y} \\ 6a^5x^3y \end{array}$$

EXPLANATION.—Since the multiplier is composed of the factors 2, a^3, x^2, and y, the multiplicand may be multiplied by each successively. 2 times $3a^2x = 6a^2x$; a^3 times $6a^2x = 6a^5x$ (Prin. 4); x^2 times $6a^5x = 6a^5x^3$ (Prin. 4); y times $6a^5x^3 = 6a^5x^3y$, since literal quantities when multiplied may be written one after another without the sign of multiplication.

The coëfficient of the product is obtained by multiplying 3 by 2 (Prin. 3). The literal quantities are multiplied by adding their exponents (Prin. 4).

Hence, the product is $6a^5x^3y$.

2. What is the product of $2a - b^2$ multiplied by $-3b$?

PROCESS.

$$\begin{array}{r} 2a \quad - \quad b^2 \\ \underline{\qquad\quad -3b} \\ -6ab + 3b^3 \end{array}$$

EXPLANATION.—Since $2a$ multiplied by $-3b$ is the same as $2a$ times $-3b$ (Prin. 1), the product of $2a$ multiplied by $-3b$ is $-6ab$. But, since the *entire* multiplicand is $2a - b^2$, the product of b^2 multiplied by $-3b$ must be *subtracted* from $-6ab$. b^2 multiplied by $-3b$ gives as a product $-3b^3$, which subtracted from $-6ab$ gives the entire product $-6ab + 3b^3$; or,

Since $2a$ and $-3b$ have *unlike* signs, the *sign* of their product is $-$ (Prin. 2); and, since $-b^2$ and $-3b$ have *like* signs, the *sign* of their product is $+$ (Prin. 2). Hence, the product is $-6ab + 3b^3$.

RULE.—*Multiply each term of the multiplicand by the multiplier, as follows:*

To the product of the numerical coëfficients, annex each literal factor with an exponent equal to the sum of the exponents of that letter in both factors.

Write the sign + before each term of the product when its factors have like signs, and — when they have unlike signs.

EXAMPLES.

	(3.)	(4.)	(5.)	(6.)	(7.)	(8.)
Multiply	-8	4	$7a$	$-3x$	$4x$	$3x^2$
By	3	-3	3	4	-5	$2x^4$

	(9.)	(10.)	(11.)	(12.)	(13.)
Multiply	$3x^3$	$4x^3y$	$3x^2y^2$	$-4xmy$	$-10xyz^2$
By	$2x^4$	$2x^2y$	$2x^3y^4$	$3x^2my^2$	$4x^2y^4z$

	(14.)	(15.)	(16.)
Multiply	$-2x^3y^2z^2$	$6abx^3$	$-5a^3b^2xy$
By	$-8xyz$	$-4a^2b^2x$	$-3a^3b^2xy$

	(17.)	(18.)	(19.)
Multiply	$-3c^2dy$	$5a^2x^2y^3$	$-6x^2y^2z^2$
By	$4c^2d$	$-3a^2x^4z$	$-4x^2y^2$

	(20.)	(21.)	(22.)	(23.)
Multiply	$4a^2x^2y^2$	$5ax^2y$	$(x+y)$	$4(a+b)$
By	$-3y^2z^2$	$-3bx^2z$	2	-3

	(24.)	(25.)	(26.)
Multiply	$-5(y+z)^2$	$(a-b)^3$	$2(c+d)^2$
By	$-3(y+z)^3$	$4(a-b)^3$	$3(c+d)^3$

	(27.)	(28.)	(29.)	(30.)
Multiply	$2(x+y+z)^4$	$3x^n$	$4a^n$	$-5ax$
By	$-5(x+y+z)^3$	$4x^n$	$-5a^{2n}$	$3a^2x^{3n}$

	(31.)	(32.)	(33.)
Multiply	$2ax^n$	$3a^nx^n$	$-4x^ny^m$
By	$4ax^m$	$-5a^{2n}x^{3n}$	$-5x^my^n$

34. Multiply $x^2 - 2y$ by $3y$.
35. Multiply $x^2y - 2z$ by $2z$.
36. Multiply $4x^2 - 2xy$ by $3xy$.
37. Multiply $-3x^2 - 2y^2$ by $2x^2y$.
38. Multiply $4x^2y^2 + 2z^2$ by $-4x^2z^2$.
39. Multiply $3x^2y^2 - 2yz$ by $3xyz$.
40. Multiply $4x^3 + 2y + 3z$ by xy.
41. Multiply $3x^2y + y - 3xz$ by $2xz$.
42. Multiply $6x^2y^2 + 4y^2 - 6z^2$ by $3xy^2$.
43. Multiply $4ab - 3ac - 3ad$ by $3acd$.
44. Multiply $5ac - 6ax + 4ab$ by $-5acx$.
45. Multiply $5abc - 3acd - 3bcd$ by $-4abcd$.
46. Multiply $3a^2xy - 2a^2bc + 4axy$ by $-2ax^2$.

CASE II.

72. To multiply when the multiplier is a polynomial.

1. Multiply $x - 2y$ by $2x + y$.

PROCESS.

$$\begin{array}{rl} & x - 2y \\ & \underline{2x + y} \\ 2x \text{ times } (x-2y) = & 2x^2 - 4xy \\ y \text{ times } (x-2y) = & \underline{\qquad\quad xy - 2y^2} \\ (2x+y) \text{ times } (x-2y) = & 2x^2 - 3xy - 2y^2 \end{array}$$

EXPLANATION.—Since the multiplier is $2x+y$, the multiplicand is to be multiplied by $2x$ and by y. $2x$ times $(x-2y)=2x^2-4xy$; y times $(x-2y)=xy-2y^2$.

Therefore, the sum of these two partial products is the entire product. Hence, the product is $2x^2-3xy-2y^2$.

RULE.—*Multiply each term of the multiplicand by each term of the multiplier, and add the partial products.*

	(2.)	(3.)
Multiply	$ab+2c$	$3x^2-axy$
By	$2ab-3c$	$2x^2+3axy$
	$2a^2b^2+4abc$	$6x^4-2ax^3y$
	$-3abc-6c^2$	$+9ax^3y-3a^2x^2y^2$
Product,	$2a^2b^2+abc-6c^2$	$6x^4+7ax^3y-3a^2x^2y^2$

4. Multiply $x+y$ by $x-y$.
5. Multiply $3a+c$ by $a+3c$.
6. Multiply $4a-2b$ by $3a-3b$.
7. Multiply $2y+3z$ by $3y-4z$.
8. Multiply $2x+y$ by $2x+2y$.
9. Multiply $3x-4y$ by $3x-4y$.
10. Multiply $5a+2c$ by $3a-7c$.
11. Multiply $ax+by$ by $ax+by$.
12. Multiply $2ac+3bc$ by $2ac-3bc$.
13. Multiply $3bd-4bc$ by $2bd+3bc$.
14. Multiply $3x^2y^2-4z^2$ by $2x^2y^2+3z^2$
15. Multiply $3x^2z^2+2y$ by $2xy^2+z$.
16. Multiply $4ab^2+3bc^2$ by $2ab+2bc^2$.
17. Multiply $5x^3y-3ax$ by $5xy^3-2ax$.
18. Multiply $a^2+2ab+b^2$ by $a+b$.
19. Multiply x^2+4x+4 by $x+2$.
20. Multiply a^2+ay-y^2 by $a-y$.
21. Multiply $2a^2+ab-2b^2$ by $3a-3b$
22. Multiply $a^6+a^4+a^2$ by a^2-1.

23. Multiply $x^4 + x^2y^2 + y^4$ by $x^2 - y^2$.
24. Multiply $2x - 3y + 4z$ by $3x + 2y - 5z$.
25. Multiply $2a^2 + 5ab - 3c^2$ by $3a^2 - 2ab + 5c^2$.
26. Multiply $3x^2 - 4xy + 5y^2$ by $7x^2 - 2xy - 3y^2$.
27. Multiply $1 - 3x + 3x^2$ by $1 - 2x + 2x^2$.
28. Multiply $a^2 + ax + x^2$ by $a^2 - ax + x^2$.
29. Multiply $x + 2y - z$ by $x + y - 2z$.
30. Multiply $a^m + b^m$ by $a^m - b^m$.
31. Multiply $x^n + y^n$ by $x^n + y^n$.
32. Multiply $x^m + y^m$ by $x^n + y^n$.
33. Multiply $x^{m+n} + y^{m+n}$ by $x^{m+n} + y^{m+n}$.
34. Multiply $a^{m-n} + b^{m-n}$ by $a^{m-n} - b^{m-n}$.

The multiplication of polynomials is sometimes *indicated* by placing them in parentheses. When the multiplication is performed, they are said to be *expanded*.

35. Expand $(x + y)(x + y)$.
36. Expand $(2x - y)(2x - y)$.
37. Expand $(3x - 4y)(3x + 4y)$.
38. Expand $(4x + 6y)(4x - 6y)$.
39. Expand $(3ax + 2y)(3ax + 2z)$.
40. Expand $(2x - 4xy)(2x - 2z)$.
41. Expand $(3a^2 - 2bc)(3a + 2bc)$.
42. Expand $(a^2 + b)(a + b^2)$.
43. Expand $(a + b + c)(a - b - c)$.
44. Expand $(a + b)(a + b)(a + b)$.
45. Expand $(a - b)(a + b)(a - b)(a + b)$.
46. Expand $(x^2 + 2x + 1)(x^2 - 2x + 1)$.
47. Expand $(a^2 - 2ab + b^2)(a^2 + 2ab + b^2)$.
48. Expand $(1 + a)(1 - a)(1 + a)(1 + a)$.
49. Expand $(x^2 - y^2)(x^2 - y^2)(x^2 - y^2)(x^2 - y^2)$.
50. Expand $(a^2 + b^2)(a^2 - b^2)(a^4 - b^4)(a^8 - b^8)$.
51. Expand $(a^2b^2 - b^2)(a^2b^2 + b^2)(a^4b^4 - b^4)(a^8b^8 - b^8)$

EQUATIONS AND PROBLEMS.

73. 1. $5(x-3)=2(x+3)+3$. Find the value of x.

PROCESS.

$$\begin{array}{lrl} & 5(x-3) &= 2(x+3)+3 \\ \text{Multiplying,} & 5x-15 &= 2x+6+3 \\ \text{Transposing,} & 5x-2x &= 15+6+3 \\ \text{Uniting,} & 3x &= 24 \\ & x &= 8 \end{array}$$

EXPLANATION.—Since the multiplication is indicated in one term on each side of the equation, in finding the value of x, the multiplication must be performed.

The known quantities are then transposed to the second member and the unknown quantities to the first member, similar terms united, and the value of x found.

VERIFICATION.—Since the value of x is 8, the substitution of 8 for x in the original equation will give an equation in which the members are identical, if the result is correct. Substituting, the original equation becomes $5(8-3)=2(8+3)+3$, which, simplified, becomes $25=25$. Hence, 8 is the value of x.

Find the value of x and verify the result in the following:

2. $3(2x-5)=21$.
3. $4+3(3x-7)=19$.
4. $3(4x+7)+5=50$.
5. $5x+3(2-x)=40$.
6. $6x+3(4x+3)=41$.
7. $5(x+6)=2(x+3)+30$.
8. $3(2x-4)=4(x-5)+32$.
9. $3(x+2)=4(x-2)+15$.
10. $3x-2(x+1)=13-7$.
11. $5x-3(x-4)=4x+7$.

12. $4(x-5)-3(x+6)=0$.
13. $(2+x)(x+3)=x^2+2x+18$.
14. $5(2x-2)=27+3(2x+1)$.
15. $10(x-5)=(x+1)+5(x+1)$.
16. $5(x+3)-2(2x-7)=3(x-7)$.
17. $3+7(x-2)-4(2x-7)=16+(x-2)$.
18. $6x=15+3(x-3)-3(x-10)$.
19. $19=2(4-x)+5(7+2x)-48$.
20. $2x+3(6x-5)-5=x-1$.
21. $3(x-7)=14+2(x-10)+2$.

Solve the following problems, and verify the result:

22. There are two numbers whose sum is 40. One is twice the other increased by 5. What are they?

PROCESS.

Let x represent the first number.
Then, $2(x+5)$ will represent the second number.
And, $x+2(x+5)=40$

$$x+2x+10=40$$
$$3x=40-10$$
$$3x=30$$
$$x=10, \text{ the first number}$$
$$2(10+5)=30, \text{ the second number}$$

23. What number is that to which, if 3 times the sum of the number and 2 is added, the result will be 22?

24. If B were 5 years younger, A's age would be twice B's. The sum of their ages is 20. How old is each?

25. Two boys find that they have together 21 cents. They discover that if Henry had 5 cents less, John's money would be just 3 times Henry's. How much has each?

26. Two pedestrians travel toward each other at the rate

of 5 miles per hour until they meet. When they meet they discover that one has traveled 3 hours longer than the other, and that the entire distance traveled by both is 55 miles. How far does each travel?

27. Three men, A, B, and C, each had a sum of money. A had twice as much as B, and B twice as much as C. A and B each lost 10 dollars and C gained 5 dollars, when the difference between what A and B had was equal to what C then had. How much had each?

28. A farmer plowed two fields containing together 50 acres. If the smaller field had contained 10 acres more, it would have been half the size of the larger. How many acres were there in each field?

29. A commenced business with twice as much capital as B. During the first year A gained $500 and B lost $300, when A had 3 times as much money as B. What was the original capital of each?

30. A man wishing to buy a quantity of butter found two firkins, one of which lacked 6 pounds of containing enough, and the other weighed 14 pounds more than he wanted. If three times the quantity in the first firkin was equal to twice the quantity in the second, how many pounds did he wish to purchase? How many pounds were there in each firkin?

SPECIAL CASES IN MULTIPLICATION.

74. By performing the operations indicated in the following examples, it is found that,

$$(a + b)(a + b) = a^2 + 2ab + b^2$$
$$(x + y)(x + y) = x^2 + 2xy + y^2$$
$$(x^2 + z^2)(x^2 + z^2) = x^4 + 2x^2z^2 + z^4$$

1. When a quantity is multiplied by itself, what power is obtained?

2. In the square of $(a+b)$ or $(x+y)$, how is the first term of the power obtained from the first term of the quantity to be squared?

3. How is the second term of the power obtained?

4. How is the third term of the power obtained?

5. What signs connect the terms of the power?

75. PRINCIPLE.—*The square of the sum of two quantities is equal to the square of the first quantity, plus twice the product of the first and second, plus the square of the second.*

EXAMPLES.

Write out the products or powers of the following:

1. $(c+d)(c+d)$.
2. $(m+n)(m+n)$.
3. $(r+s)(r+s)$.
4. $(x+2)(x+2)$.
5. $(a+3)(a+3)$.
6. $(3a+x)(3a+x)$.
7. Square $2x+4y$.
8. Square $3a+2b$.
9. Square x^2+y^2.
10. Square $4x+3y$.
11. Square $3p+2q$.
12. Square $2x^2+5y^2$.

76. By performing the operations indicated in the following examples, it is found that,

$$(a-b)(a-b)=a^2-2ab+b^2$$
$$(x-y)(x-y)=x^2-2xy+y^2$$
$$(x^2-z^2)(x^2-z^2)=x^4-2x^2z^2+z^4$$

1. How are the terms of the power obtained from the terms of the quantity squared?

2. What signs connect the terms of the power?

3. How does the square of $(a-b)$ differ from the square of $(a+b)$?

77. PRINCIPLE.—*The square of the difference of two quantities is equal to the square of the first quantity, minus twice the product of the first and second, plus the square of the second.*

EXAMPLES.

Write out the products or powers of the following:

13. $(a-c)(a-c)$.
14. $(y-z)(y-z)$.
15. $(r-s)(r-s)$.
16. $(b-c)(b-c)$.
17. $(x-1)(x-1)$.
18. $(x-2y)(x-2y)$.
19. Square $a-d$.
20. Square $2r-3s$.
21. Square $2s-q$.
22. Square $3m-4n$.
23. Square $2v-w$.
24. Square $2x^2-2y^2$.

78. By performing the operations indicated in the following examples, it is found that,

$$(a+b)(a-b)=a^2-b^2$$
$$(x-y)(x+y)=x^2-y^2$$
$$(x^2+z^2)(x^2-z^2)=x^4-z^4$$

1. How are the terms of the product obtained from the quantities?
2. What sign connects the terms?

79. PRINCIPLE.—*The product of the sum and the difference of two quantities is equal to the difference of their squares.*

EXAMPLES.

25. $(c+d)(c-d)$.
26. $(r+s)(r-s)$.
27. $(m+n)(m-n)$.
28. $(c+a)(c-a)$.
29. $(x-1)(x+1)$.
30. $(2-x)(2+x)$.
31. $(2x+4)(2x-4)$.
32. $(2x^2+y)(2x^2-y)$.
33. $(x^2+y^2)(x^2-y^2)$.
34. $(x^4-y^4)(x^4+y^4)$.
35. $(3v+2w)(3v-2w)$.
36. $(5xy-3)(5xy+3)$.

80. By performing the operations indicated in the following examples, it is found that,

$$(x+2)(x+3)=x^2+5x+6$$
$$(x+2)(x-3)=x^2-x-6$$
$$(x-2)(x-3)=x^2-5x+6$$

1. How many terms are alike in each factor?

2. How is the first term of each product obtained from the factors?

3. How is the second term of the product in the first example obtained from the factors? In the second example? In the third example?

4. How is the third term of the product in each example obtained from the factors?

5. How are the signs determined which connect the terms?

81. PRINCIPLE.—*The product of two binomial quantities having a common term is equal to the square of the common term, the algebraic sum of the other two multiplied by the common term, and the algebraic product of the unlike terms.*

EXAMPLES.

Write out the products of the following:

37. $(x+4)(x+3)$.
38. $(x-5)(x+3)$.
39. $(x+3)(x-4)$.
40. $(x-4)(x-6)$.
41. $(a+3)(a+b)$.
42. $(a+m)(a+n)$.
43. $(2x+4)(2x-5)$.
44. $(3x-7)(3x+5)$.
45. $(2y-3)(2y-4)$.
46. $(4a+b)(4a+c)$.
47. $(5a+2b)(5a-2c)$.
48. $(3ax+4)(3ax-7)$.
49. $(2a^2x+2)(2a^2x-6)$.
50. $(2x^2y^3+4)(2x^2y^3+7)$.

DIVISION.

INDUCTIVE EXERCISES.

82. 1. How many times are 2 bushels contained in 8 bushels? $2b$ in $8b$? $2a$ in $8a$? 3 bushels in 12 bushels? $3b$ in $12b$?

2. How many times is $5d$ contained in $10d$? $5x$ in $20x$? $6y$ in $12y$?

3. How is the coëfficient of the quotient determined?

4. What is the product of $a^2 \times a^3$?

5. Since the product of $a^2 \times a^3$ is a^5, if a^5 is divided by a^2 what will be the quotient? What will be the quotient if a^5 is divided by a^3?

6. What is the product when x^2 is multiplied by x^4?

7. What is the exponent of the quotient when x^6 is divided by x^2? By x^4? x^9 by x^3? x^9 by x^6? x^5 by x^4? x^5 by x?

8. How is the exponent of a quantity in the quotient determined?

9. When $+5$ is multiplied by $+3$, what is the product?

10. Since $+15$ is the product of $+5 \times +3$, if $+15$ is divided by $+3$ what is the sign of the quotient? What when $+15$ is divided by $+5$?

11. What is the sign of the quotient when a positive quantity is divided by a positive quantity?

12. When $+5$ is multiplied by -3, what is the product?

13. Since -15 is the product of $+5 \times -3$, if -15

is divided by $+5$ what is the sign of the quotient? What when it is divided by -3?

14. What is the quotient of -24 divided by -4? By $+4$? By $+6$? By -3? By $+3$? By -8? By $+8$?

15. What is the sign of the quotient when a negative quantity is divided by a positive quantity?

16. What is the sign of the quotient when a negative quantity is divided by a negative quantity?

17. What is the product of -4 by -3?

18. Since $+12$ is the product of -4×-3, if $+12$ is divided by -3 what is the sign of the quotient? What when it is divided by -4?

19. What is the quotient of $+24$ divided by -3? By -4? By -6? By -8? By -12?

20. What is the sign of the quotient when a positive quantity is divided by a negative quantity?

DEFINITIONS.

83. **Division** is the process of finding how many times one quantity is contained in another. Or,

The process of separating a quantity into equal parts.

84. The **Dividend** is the quantity to be divided.

85. The **Divisor** is the quantity by which we divide. It shows into how many equal parts the dividend is to be divided.

86. The **Quotient** is the result obtained by division. The part of the dividend remaining when the division is not exact is called the **Remainder**.

87. The **Signs of Division.** (See Art. 14.)

88. Principles.—1. *The sign of any term of the quotient is + when the dividend and divisor have like signs, and — when they have unlike signs.*

2. *The coëfficient of the quotient is equal to the coëfficient of the dividend divided by that of the divisor.*

3. *The exponent of any quantity in the quotient is equal to its exponent in the dividend diminished by its exponent in the divisor.*

89. The principle relating to the signs in division may be illustrated as follows:

$$\left.\begin{array}{l} +a \times +b = +ab \\ -a \times +b = -ab \\ +a \times -b = -ab \\ -a \times -b = +ab \end{array}\right\} \text{Hence, } \left\{\begin{array}{l} +ab \div +b = +a \\ -ab \div +b = -a \\ -ab \div -b = +a \\ +ab \div -b = -a \end{array}\right.$$

CASE I.

90. To divide when the divisor is a monomial.

1. Divide $-15x^2y^3z^4$ by $3xy^2z^2$.

PROCESS.

$$3xy^2z^2)\underline{-15x^2y^3z^4}$$
$$-5xyz^2$$

Explanation.—Since the dividend and divisor have unlike signs, the sign of the quotient is —. (Prin. 1.)

Then -15 divided by 3 is -5; x^2 divided by x is x; y^3 divided by y^2 is y; and z^4 divided by z^2 is z^2 (Prin. 3). Therefore, the quotient is $-5xyz^2$.

2. Divide $12a^2x^2y^3$ by $5a^2x^2z^2$.

PROCESS.

$$\frac{12a^2x^2y^3}{5a^2x^2z^2} = \frac{12a^2x^2y^3}{5a^2x^2z^2} = \frac{12y^3}{5z^2}$$

Explanation.—Since division may be indicated by writing the divisor under the dividend with a line between

them, we have $\frac{12a^2x^2y^3}{5a^2x^2z^2}$. And since the same factors are found in both dividend and divisor, they may be cancelled without changing the quotient.

Hence, the quotient is $\frac{12y^3}{5z^2}$.

3. Divide $9a^2x^3 - 12a^3x^5 + 6ax^4$ by $3ax^2$.

PROCESS.

$$3ax^2\overline{)\,9a^2x^3 - 12a^3x^5 + 6ax^4}$$
$$3ax - 4a^2x^3 + 2x^2$$

EXPLANATION.—Dividing $9a^2x^3$ by $3ax^2$, the result is $3ax$, dividing $-12a^3x^5$ by $3ax^2$, the result is $-4a^2x^3$; dividing $6ax^4$ by $3ax^2$, the result is $2x^2$.

Therefore, the quotient is $3ax - 4a^2x^3 + 2x^2$.

RULE.—*Divide each term of the dividend by the divisor, as follows:*

To the numerical coëfficient of the dividend divided by that of the divisor, annex each literal factor with an exponent equal to the exponent of that letter in the dividend minus its exponent in the divisor.

Write the sign + before each term of the quotient when the terms of both dividend and divisor have like signs, and — when they have unlike signs.

PROOF.—*The same as in Arithmetic.*

1. An equal factor in both dividend and divisor is omitted from the quotient.

2. When the division is not exact, the common factors should be cancelled and the remaining factors written as a fraction.

	(4.)	(5.)	(6.)	(7.)	(8.)
Divide	$6a$	$-12a^2x$	$15a^2y^2$	$-20x^3y^2$	$24y^2z^3$
By	$3a$	$3a^2x$	$-5ay$	$-5x^2y$	$-8y^2z$

9. Divide $28x^2y^2z^2$ by $7xyz$.

Find the quotient in the following, and prove:

10. $-25x^2y^2z^3 \div 5xyz^2$.
11. $20a^5b^5c \div 10abc$.
12. $30cd^2f \div 15cd^2$.
13. $36ax^2y \div 18ay$.
14. $-18x^2yz \div 9xy$.
15. $-21vwz^2 \div 7vz^2$.
16. $-33r^2sz^2 \div 11rs$.
17. $35m^2nx \div 5m^2x$.
18. $20x^3y^3z^3 \div 10x^3yz^3$.
19. $-14a^2x^3y^4 \div 7axy^2$.
20. $32r^2s^2q \div 8r^2sq$.
21. $-18v^2x^2y \div v^2xy$.
22. $24a^{2n}bc^n \div -a^{2n}bc^n$.
23. $36n^{2n}xy^{3n} \div -4n^nxy^{2n}$.
24. $25xyz^2 \div -5x^2y^2z$.
25. $-28y^2z^{2m} \div 4y^2z^3x$.
26. $-30n^2x^2 \div 6m^2x^2$.
27. Divide $2(x+y)$ by 2.
28. Divide $3a(x+y)$ by 3.
29. Divide $-12c(x+z)$ by $4c$.
30. Divide $20(y+z)$ by $(y+z)$.
31. Divide $18(x+z)^5$ by $(x+z)^2$.
32. Divide $-8a(x-y)^3$ by $-8(x-y)^2$.
33. Divide $5a^2(c+d)^5$ by $-2a(c+d)^2$.
34. Divide $10x^2y(y-z)^4$ by $-5y(y-z)^4$.
35. Divide ax^2y-2xy^2 by xy.
36. Divide $3xy^2-3x^2y$ by xy.
37. Divide $4x^3y^2+2x^2y^3$ by $2x^2y^2$.
38. Divide $3a^2b^2-6ab^3$ by $3ab$.
39. Divide $abc^2-a^2b^2c$ by $-abc$.
40. Divide $9x^2y^2z+3xyz^2$ by $3xyz$.
41. Divide $15ax^2y^2-5a^2x^2y$ by $-5axy$.
42. Divide $30x^3y^3z^3-20x^2y^2z^2$ by $-5x^2y^2z^2$.
43. Divide $a^2-3ab+ac^2$ by a.
44. Divide $x^2y-xy^2+x^2y^3$ by xy.
45. Divide $x^2-2xy+y^2$ by x.
46. Divide $z^2-3xz+3z^2$ by z.
47. Divide $m^2n+2mn-3m^2$ by mn.
48. Divide $c^2d-3cd^2+4d^3$ by cd.
49. Divide $a^2x-3a^2y+3a^2x^2$ by a^2x.

50. Divide $v^2y - 2y^2 + 3v^2y^2$ by vy.
51. Divide $3(x+y) - 2(x+y)^2$ by $-(x+y)$.
52. Divide $a(b+c)^2 + b(b+c)^3$ by $-(b+c)$.
53. Divide $9(a-c) - 6(a-c)^3$ by $3(a-c)$.
54. Divide $5(x+z)^3 - 10(x+z)^4$ by $-5(x+z)$.

CASE II.

91. To divide when the divisor is a polynomial.

1. Divide $x^3 + 3x^2y + 3xy^2 + y^3$ by $x + y$.

PROCESS.

$$
\begin{array}{l|l}
x^3 + 3x^2y + 3xy^2 + y^3 & x + y \\
\cline{2-2}
\underline{x^3 + x^2y} & x^2 + 2xy + y^2 \\
\qquad 2x^2y + 3xy^2 & \\
\qquad \underline{2x^2y + 2xy^2} & \\
\qquad\qquad\quad xy^2 + y^3 & \\
\qquad\qquad\quad \underline{xy^2 + y^3} &
\end{array}
$$

EXPLANATION.—For convenience, the divisor is written at the right of the dividend, both of which are arranged according to the descending powers of x.

Since the first term of the dividend is equal to the product of the first term of the divisor by the first term of the quotient, if the first term of the dividend is divided by the first term of the divisor, the result will be the first term of the quotient. x is contained in x^3, x^2 times; therefore, x^2 is the first term of the quotient. x^2 times the divisor equals $x^3 + x^2y$. Subtracting, there is a remainder of $2x^2y$, to which the next term of the dividend is annexed for a new dividend.

Since the first term of the new dividend is the product of the first term of the divisor by the second term of the quotient, if the first term of the new dividend is divided by the first term of the divisor, the result will be the second term of the quotient. x is contained in $2x^2y$, $2xy$ times; therefore, $2xy$ is the second term

of the quotient. $2xy$ times the divisor equals $2x^2y+2xy^2$. Subtracting, there is a remainder of xy^2, to which the next term of the dividend is annexed for a new dividend.

Reasoning as before, the third term of the quotient is found by dividing the first term of the new dividend by the first term of the divisor. x is contained in xy^2, y^2 times. y^2 times the divisor is xy^2+y^3. Subtracting, there is no remainder. Hence, the quotient is $x^2+2xy+y^2$.

RULE.—*Write the divisor at the right of the dividend, arranging the terms of each according to the ascending or descending powers of one of the literal quantities.*

Divide the first term of the dividend by the first term of the divisor, and write the result for the first term of the quotient.

Multiply the divisor by this term of the quotient, subtract the product from the dividend, and to the remainder annex as many terms of the dividend as are necessary to form a new dividend.

Divide the new dividend as before, and continue to divide in this way until the first term of the divisor is not contained in the first term of the dividend.

If there be a remainder after the last division, write it over the divisor in the form of a fraction, and annex it to the quotient with its proper sign.

PROOF.—*To the product of the quotient and divisor add the remainder, if any, and the sum will be equal to the dividend, if the work is correct.*

2.
$$\begin{array}{llll|l}
x^4 & -a^2x^2 & +2a^3x & -a^4 & x^2+ax-a^2 \\ \cline{5-5}
x^4+ax^3 & -a^2x^2 & & & x^2-ax+a^2 \\ \cline{1-3}
-ax^3 & & +2a^3x & -a^4 & \\
-ax^3 & -a^2x^2 & +a^3x & & \\ \cline{1-3}
 & a^2x^2 & +a^3x & -a^4 & \\
 & a^2x^2 & +a^3x & -a^4 & \\ \cline{2-4}
\end{array}$$

3.
$$\begin{array}{l|l}
x^4 - 1 & x - 1 \\
\underline{x^4 - x^3} & \overline{x^3 + x^2 + x + 1} \\
\quad x^3 - 1 & \\
\quad \underline{x^3 - x^2} & \\
\qquad x^2 - 1 & \\
\qquad \underline{x^2 - x} & \\
\qquad\quad x - 1 & \\
\qquad\quad \underline{x - 1} &
\end{array}$$

4.
$$\begin{array}{l|l}
a^3 - x^3 & a - x \\
\underline{a^3 - a^2x} & \overline{a^2 + ax + x^2} \\
\quad a^2x - x^3 & \\
\quad \underline{a^2x - ax^2} & \\
\qquad ax^2 - x^3 & \\
\qquad \underline{ax^2 - x^3} &
\end{array}$$

5.
$$\begin{array}{l|l}
48x^3 - 76ax^2 - 64a^2x + 105a^3 & 2x - 3a \\
\underline{48x^3 - 72ax^2} & \overline{24x^2 - 2ax - 35a^2} \\
\quad - 4ax^2 - 64a^2x & \\
\quad \underline{- 4ax^2 + 6a^2x} & \\
\qquad - 70a^2x + 105a^3 & \\
\qquad \underline{- 70a^2x + 105a^3} &
\end{array}$$

6. Divide $a^2 - 2ab + b^2$ by $a - b$.
7. Divide $x^2 + 4x + 4$ by $x + 2$.
8. Divide $9 + 6x + x^2$ by $3 + x$.
9. Divide $x^3 + x^2y + xy^2 + y^3$ by $x + y$.
10. Divide $a^4 + a^3y + ay^3 + y^4$ by $a + y$.
11. Divide $x^3 + 3x^2y + 3xy^2 + y^3$ by $x + y$.
12. Divide $r^3 + 3r^2s + 3rs^2 + s^3$ by $r^2 + 2rs + s^2$.
13. Divide $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$ by $x + y$.
14. Divide $c^4 + 4c^3d + 6c^2d^2 + 4cd^3 + d^4$ by $c^2 + 2cd + d^2$.
15. Divide $x^4 - 3x^3 - 36x^2 - 71x - 21$ by $x^2 - 8x - 3$.

16. Divide $a^3 + 5a^2x + 5ax^2 + x^3$ by $a^2 + 4ax + x^2$.
17. Divide $a^2 + 2bc - b^2 - c^2$ by $a - b + c$.
18. Divide $a^4 - 4a^3y + 6a^2y^2 - 4ay^3 + y^4$ by $a^2 - 2ay + y^2$.
19. Divide $ax^3 - a^2x^2 - bx^2 + b^2$ by $ax - b$.
20. Divide $20a^2b - 25a^3 - 18b^3 + 27ab^2$ by $6b - 5a$.
21. Divide $3x^4 - 8x^2y^2 + 3x^2z^2 + 5y^4 - 3y^2z^2$ by $x^2 - y^2$.
22. Divide $4a^4 - 9a^2 + 6a - 1$ by $2a^2 + 3a - 1$.
23. Divide $2ay + 3by + 10ab + 15b^2$ by $y + 5b$.
24. Divide $b - 6b^3 - 2a + 54a^3 - 3a^2b$ by $2a - b$.
25. Divide $25a^5 - a^3 - 8a - 2a^2$ by $5a^2 - 4a$.
26. Divide $x^3 + y^3 + z^3 - 3xyz$ by $x + y + z$.
27. Divide $18x^4 - 45x^3 + 82x^2 - 67x + 40$ by $3x^2 - 4x + 5$.
28. Divide $16x^4 - 72a^2x^2 + 81a^4$ by $2x - 3a$.
29. Divide $a^4 + 4a^2x^2 + 16x^4$ by $a^2 + 2ax + 4x^2$.
30. Divide $x^4 + x^2z^2 + z^4$ by $x^2 - xz + z^2$.
31. Divide $x^4 - y^4$ by $x - y$.
32. Divide $x^5 + y^5$ by $x + y$.
33. Divide $x^7 + 1$ by $x + 1$.
34. Divide $x^4 - 81y^4$ by $x - 3y$.
35. Divide $81a^4 - 16b^4$ by $3a + 2b$.
36. Divide $x^n + y^n$ by $x + y$ to 3 terms.

ZERO AND NEGATIVE EXPONENTS.

92. 1. How many times is a^2 contained in a^2? a^3 in a^3? a^4 in a^4? a^8 in a^8? a^m in a^m?

2. When similar quantities have exponents, how may the division be performed?

3. What, then, will be the quotient when a^2 is divided by a^2 by subtracting exponents? a^8 by a^3? a^4 by a^4? a^{20} by a^{20}? a^m by a^m?

4. Since $a^2 \div a^2$, $a^3 \div a^3$, $a^4 \div a^4$, and $a^m \div a^m$ are each

equal to a^0 and to 1, what is the value of a^0, or any quantity with 0 for an exponent?

5. What will be the quotient when a^3 is divided by a^5 by subtracting exponents? a^5 by a^7? a^6 by a^9?

6. What will be the quotient when a^3 is divided by a^5 without subtracting exponents? a^5 by a^7? a^6 by a^9?

7. Since $a^3 \div a^5 = a^{-2}$ and $\frac{1}{a^2}$, $a^5 \div a^7 = a^{-2}$ and $\frac{1}{a^2}$, $a^6 \div a^9 = a^{-3}$ and $\frac{1}{a^3}$, to what is any quantity with a *negative* exponent equal?

The **Reciprocal** of a quantity is 1 divided by the quantity.

Thus the reciprocal of a is $\frac{1}{a}$, of $x + y$ is $\frac{1}{x+y}$.

93. PRINCIPLES.—1. *Any quantity having 0 for an exponent is equal to 1.*

2. *Any quantity having a negative exponent is equal to the reciprocal of the quantity with an equal positive exponent.*

1. Divide abx by $-abx$.
2. Divide $6a^2x^3$ by $3a^2x^2$.
3. Divide $8a^3x^5$ by $-4ax^5$.
4. Divide $27x^3y^2z^2$ by $-9x^2y^2z$.
5. Divide $30x^5(y+z)^2$ by $5x^2(y+z)^2$.
6. What is the reciprocal of x^{-2}?
7. What is the reciprocal of $x^{-2}y^{-2}$?
8. What is the reciprocal of $a^2x^{-2}y^{-3}$?
9. Divide $12x^{-3}y^{-4}$ by $x^{-5}y^7$.
10. Divide $20a^{-4}b^3c^{-3}$ by $-5a^{-2}b^6c^{-5}$.
11. What is the reciprocal of $2x^{-3}y^{-2}$?
12. What is the reciprocal of $3x^{-m}y^{-n}$?

EQUATIONS AND PROBLEMS.

94. 1. Find the value of x in the equation $ax + 4 = a^2 - 2x$.

PROCESS.

$$
\begin{array}{ll}
 & ax + 4 = a^2 - 2x \\
\text{Transposing,} & ax + 2x = a^2 - 4 \\
\text{Then,} & (a + 2)x = a^2 - 4 \\
\text{And,} & x = a - 2
\end{array}
$$

EXPLANATION.—Transposing the known quantities to one member and the unknown to the other, the coefficient of x is $a + 2$.

Dividing both members of the equation by $a + 2$, the value of x is found to be $a - 2$.

2. Find the value of x in the equation $bx - b^2 = 4x - 9b + 20$.

PROCESS.

$$
\begin{array}{ll}
 & bx - b^2 = 4x - 9b + 20 \\
\text{Transposing,} & bx - 4x = b^2 - 9b + 20 \\
\text{Then,} & (b - 4)x = b^2 - 9b + 20 \\
\text{And,} & x = b - 5
\end{array}
$$

EXPLANATION.—Transposing as before, it is seen that the coëfficient of x is $b - 4$.

Dividing both members by $b - 4$, the coëfficient of x, the value of x is found to be $b - 5$.

Find the value of x in the following:

3. $cx - 9 = c^2 + 6c - 3x$.
4. $ax + 16 = a^2 - 4x$.
5. $3x - 12a = 4a^2 - 2ax + 9$.
6. $dx + 9a^2 = d^2 - 3ax$.
7. $ax - a^2 = 2ab + b^2 - bx$.
8. $ax - 5ab = 2a^2 + 3b^2 - bx$.

9. $ax - c^2 = a^3 + ac + a^2c - cx.$
10. $2ax - 6a^2 = 13ab + 6b^2 - 3bx.$
11. $2ax - 10ab - 15b = 14a + 21 - 3x.$
12. $ax + bx = 5a^2 + 7ab + 2b^2 + 5ac + 2bc - cx.$
13. $2cx - 4c^3 + d^2 = 2c^2d - 2cd - dx.$
14. $b^2x + 3b^2c + 6c^3 = b^3 + 2bc^2 - 2c^2x.$
15. $4m^4 - 2m^2x - 3mx = 1 - 6m + 9m^2 - x.$
16. $a^3 + 3x - 9a^2 = ax - 27a + 27.$
17. $2m^2x + 3mn^3 + 7m^2n^2 - 4m^4 = 3mnx.$
18. $5ax = 15a^3 - 5ab + 5ab^2 + 2bx - 6a^2b + 2b^2 - 2b^3.$

19. A man being asked how much money he had, replied that if he had \$25 more than 3 times what he then had, he would have \$355. How much money had he?

20. A gentleman divided \$10500 among four sons, giving to the second twice as much as to the first, to the third twice as much as to the second, and to the fourth one-half as much as to the other three. How much did he give to each?

21. A man who met some beggars gave 3 cents to each and had 4 cents left, but found that he lacked 6 cents of having enough to give them 5 cents each. How many beggars were there? How much money did he have?

22. A man has six sons, each 4 years older than the one next to him. The eldest is 3 times as old as the youngest. What is the age of each?

23. A vessel containing some water was filled by pouring into it 42 gallons, and there was then in the vessel 7 times as much as at first. How many gallons did the vessel contain at first?

24. A man borrowed as much money as he had and spent a dollar; he then borrowed as much as he had left and spent a dollar; again he borrowed as much as he then had and spent a dollar, when he had nothing left. How much had he at first?

REVIEW EXERCISES.

95. 1. Add $6ax - 140 + 3\sqrt{x}$, $5x^2 + 4ax + 9x^2$, $7ax + 4\sqrt{x} + 160$, and $\sqrt{x} + 3ax - 4x^2$.

2. Add $3am + 2x - 3\sqrt{y} - z$, $2\sqrt{y} + 3z - 2x^2 + 3am$, $4x^2 - 3z + 2\sqrt{y} + 3x$, and $2\sqrt{y} - 4am + 2z - 3x^2$.

3. From $\sqrt{a^2 - b^2} - 2(x + y) - 6$ subtract $4(x + y) - 3\sqrt{a^2 - b^2}$.

4. From $\sqrt{x} + 2\sqrt{y} - z + 6$ subtract $3\sqrt{y} - 2\sqrt{x} - y + 2z - 16$.

5. From $a^2x^2 + 2ay - 3y^5 + z^6$ subtract $b^2x^2 + 3ay - cy^5 + 4z^6$.

6. From the sum of $x^{2n} + 3x^2y^n - 3yz + az$ and $4x^n - 3yz + 2z + 3x^2y^n$ subtract $5x^{2n} - 4z + 6x^2y^2 - 3az$.

7. Multiply $x^4 + 2x^2y + xy^3$ by $x^2 + 2xy - y^2$.

8. Multiply $x^n + 2x^ny^n + y^n$ by $x^n + 2x^ny^n + y^n$.

9. Multiply $3x^{-n} + 2x^{-2n}y^{-2n} - y^n$ by $x^n - y^{2n} + x^ny^n$.

10. Multiply $3x^{n+2} + 2y^{n+m} + z^m$ by $3x^{-2} - 2y^{-n-m} + z^{2m}$.

11. Expand $(x + y)(x + y)(x + y)(x + y)(x + y)$.

12. Expand $(a + 2)(a + 2)(a - 2)(a - 2)$.

13. Expand $(3a - 6)(3a - 6)(3a + 6)(3a + 6)$.

14. Expand $(x + 2y)(x + 2y)$.

15. Square $2x + 5y$.

16. Square $3x^2 - 2y^2$.

17. Square $x^{2n} + 2y^{2n}$.

18. Square $x^{-2n} - 2y^{-2n}$.

19. Write out the product of $(2x + y)(2x - y)$.

20. Write out the product of $(3x + 7y)(3x - 7y)$.

21. Write out the product of $(4x^2 - 2y^2)(4x^2 + 2y^2)$.

22. Write out the product of $(ax^n + y^n)(ax^n - y^n)$.

23. Write out the product of $(ax^{-n} + ay^{-n})(ax^{-n} - ay^{-n})$.

24. Write out the product of $(a+x)(a-x)(a^2+x^2)(a^4+x^4)$.

25. Write out the product of $(x^2+y^2)(x^2-y^2)(x^4+y^4)(x^8+y^8)$.

26. Write out the product of $(2x+3)(2x-3)(4x^2+9)(16x^4-81)$.

27. Divide $4a^4-5a^2b^2+b^4$ by $2a^2-3ab+b^2$.

28. Divide $5x^2y+x^3+y^3+5xy^2$ by x^2+y^2+4xy.

29. Divide $x^5-5x^4y+10x^3y^2-10x^2y^3+5xy^4-y^5$ by x^2+y^2-2xy.

30. Divide $2a^{3n}-6a^{2n}b^n+6a^nb^{2n}-2b^{3n}$ by a^n-b^n.

31. Express in its equivalent without negative exponents $x^{-5}y^{-3}z^{-4}$.

32. Express in its equivalent without negative exponents $x^2y^{-3}z^{-4}$.

33. Express in its equivalent without negative exponents $a^{-2}y^2z^2$.

34. Express in its equivalent without negative exponents $r^5s^{-4}z^{-5}$.

35. Divide by subtracting exponents a^2b^3y by a^2b^5y.

36. Divide by subtracting exponents $a^4x^4y^2$ by $a^5x^3y^3$.

37. Divide by subtracting exponents $-x^{-n}y^nz^m$ by $x^{-2n}y^nz^{2m}$.

38. Divide $x^6y^0+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+x^0y^6$ by $x^2+2xy+x^0y^2$.

39. Divide $x^{-5}+5x^{-4}y^{-1}+10x^{-3}y^{-2}+10x^{-2}y^{-3}+5x^{-1}y^{-4}+y^{-5}$ by $x^{-1}+y^{-1}$.

40. Find the value of x in the equation $2ax+12ab-4a^2=9b^2+3bx$.

41. Find the value of x in the equation $3x-9-3c=12a-2ax+4a^2+2ac$.

42. Find the value of x in the equation $2ax+9c^2+3cd=4a^2+3cx+2ad$.

FACTORING.

96. 1. What is the product of 4 times $5a$? What are 4 and $5a$ of their product?

2. What is $3a$ of $9a$? $5a$ of $15a$? $3c$ of $15c$?

3. What quantity will exactly divide $10c$? $18d$? $25x$? $30z$?

4. What are the exact divisors of $12xy$? $25x^2y^2$? $36xy^2$?

5. What are the exact divisors of $24x^2y^2$? $30xy^2$? $44a^2bc$?

DEFINITIONS.

97. An **Exact Divisor** of a quantity is a quantity that will divide it without a remainder.

Thus, a, b, and $x + y$ are exact divisors of $ab(x + y)$.

98. The **Factors** of a quantity are the quantities which, being multiplied together, will produce the quantity.

Thus, a, b, and $x + y$ are the factors of $ab(x + y)$.

An *exact divisor* of a quantity is a factor of it.

99. A **Prime Quantity** is a quantity that has no exact divisor except itself and 1.

100. A **Prime Factor** is a factor that is a prime quantity.

101. **Factoring** is the process of separating a quantity into its factors.

CASE I.

102. To separate a monomial into its factors.

1 What are the prime factors of $24x^2y^3z$?

PROCESS.

$$\begin{aligned} 24 &= 2, 2, 2, 3 \\ x^2 &= xx \\ y^3 &= yyy \\ z &= z \\ 24x^2y^3z &= 2, 2, 2, 3, x, x, y, y, y, z \end{aligned}$$

EXPLANATION.—The prime factors of 24 are 2, 2, 2, and 3; of x^2, x and x; of y^3, y, y, and y; and z is a prime quantity. Therefore, the prime factors are 2, 2, 2, 3, x, x, y, y, y, z.

RULE.—*Separate the numerical coëfficient into its prime factors.*

Separate the literal quantities into their prime factors by writing each quantity as a factor as many times as there are units in its exponent.

Find the prime factors of the following:

2. $8a^2b$.	4. $15a^3y^2z$.	6. $42axy^3$.	8. $28a^2c^2x$.
3. $10x^2y^3$.	5. $20ax^3y$.	7. $36xy^2z^3$.	9. $35x^2z^2c^3$.

CASE II.

103. To separate a polynomial into monomial and polynomial factors.

1. What are the factors of $5a^2bc + 10a^2c - 20a^2bc$?

PROCESS.

$$5a^2c)\underline{5a^2bc + 10a^2c - 20a^2bc}$$
$$b + 2 - 4b$$
$$5a^2c\,(b + 2 - 4b)$$

EXPLANATION.—By examining the terms of the polynomial it is found that $5a^2c$ is a factor of every term. Dividing by this common factor the other is found. Hence, the factors are $5a^2c$ and $(b + 2 - 4b)$.

RULE.—*Divide the polynomial by the greatest factor common to all the terms. The divisor, and the quotient, will be the factors sought.*

Find the factors of the following polynomials:

2. $5a^2b + 6a^2c$.
3. $8x^2y^2 + 12x^2z^2$.
4. $6xyz + 12x^2y^2z$.
5. $9x^3y^2z + 18xy^2z^3$.
6. $a^2x^2y^2z + a^2xyz^2$.
7. $a^2c + b^2c + c^2d^2$.
8. $4x^2y + cxy^2 + 3xy^3$.
9. $a^3yz + a^2xz + a^2x^2y^2z^2$.
10. $b^2x^2y^2 + b^3xy^2 + bx^2y^3z$.
11. $a^2x^ny^nz + ax^nyz^n + a^2x^ny^2z^2$.

CASE III.

104. To separate a trinomial into two equal binomial factors.

1. When $a + b$ is multiplied by $a + b$, what is the product?
2. When $x + y$ is multiplied by $x + y$, what is the product?
3. When $c - d$ is multiplied by $c - d$, what is the product?
4. When $x - 1$ is multiplied by $x - 1$, what is the product?
5. When $x + y$ is squared, what terms are squares? To what is the other term equal?
6. When $x - 1$ is squared, what terms are squares? To what is the other term equal?

3x
3x,
9x

7. When a trinomial consists of two terms that are squares, and the third term is twice the product of the square roots of the squares, how will the factors compare?

105. One of the two equal factors of a quantity is called its *Square Root*.

106. 1. Resolve $x^2 - 2xy + y^2$ into two equal binomial factors.

PROCESS.

$$x^2 - 2xy + y^2$$
$$\sqrt{x^2} = x$$
$$\sqrt{y^2} = y$$
$$(x - y)(x - y)$$

EXPLANATION.—Since the trinomial contains two terms that are squares, and the other term is twice the product of their square roots, the quantity may be separated into two binomial factors. Since the terms that are squares are the squares of the two quantities, the square root of x^2 and of y^2 gives us x and y, the two quantities; and since twice their product has the minus sign, the quantities must have had unlike signs.

Therefore, the factors are $x - y$ and $x - y$.

RULE.—*Find the square roots of the terms that are squares, and connect these roots by the sign of the other term. The result will be one of the equal factors.*

Find the equal factors of the following trinomials:

2. $a^2 + 2ab + b^2$.
3. $x^2 + 2xy + y^2$.
4. $b^2 - 2bc + c^2$.
5. $r^2 + 2rs + s^2$.
6. $x^2 + 2x + 1$.
7. $x^2 + 4x + 4$.
8. $y^2 - 2y + 1$.
9. $4y^2 - 4y + 1$.
10. $9x^2 + 6x + 1$.
11. $9m^2 + 18mn + 9n^2$.
12. $9 + 6x + x^2$.
13. $1 - 2x^2 + x^4$.
14. $16n^2 - 8n + 1$.
15. $16 + 16a + 4a^2$.
16. $36 + 12a^2 + a^4$.
17. $49 - 14x^3 + x^6$.
18. $81x^2 - 18ax + a^2$.
19. $4a^{2n} + 12a^n b^n + 9b^{2n}$.

CASE IV.

107. To resolve a binomial into two binomial factors.

1. When $a+b$ is multiplied by $a-b$, what is the product?

2. When $x+y$ is multiplied by $x-y$, what is the product?

3. When $c+d$ is multiplied by $c-d$, what is the product?

4. When $x+2$ is multiplied by $x-2$, what is the product?

5. When the sum of two quantities is multiplied by their difference, what is the product?

6. When a binomial consists of two terms that are squares, connected by the minus sign, into what factors may it be resolved?

1. Resolve x^2-y^2 into its factors.

PROCESS.

$$x^2-y^2$$
$$\sqrt{x^2}=x$$
$$\sqrt{y^2}=y$$
$$(x+y)(x-y)$$

EXPLANATION.—Since the binomial consists of two terms that are squares connected by the minus sign, the binomial may be separated into two binomial factors, which are respectively the sum and the difference of the quantities.

The square root of x^2 is x, and of y^2 is y.
Therefore, $x+y$ and $x-y$ are the factors.

RULE.—*Find the square root of each term of the binomial, and make the sum of these square roots one factor, and their difference the other.*

Binomials of the form of x^4-y^4 may be resolved into the factors $(x^2+y^2)(x^2-y^2)$, and x^2-y^2 into $(x+y)(x-y)$. Therefore, $x^4-y^4=(x^2+y^2)(x+y)(x-y)$.

Resolve the following binomials into their factors:

2. $a^2 - b^2$.
3. $c^2 - d^2$.
4. $m^2 - n^2$.
5. $4x^2 - 4y^2$.
6. $9x^2 - y^2$.
7. $x^2 - 9y^2$.
8. $16x^2 - 16y^2$.
9. $\frac{1}{4}x^2 - \frac{1}{9}y^2$.
10. $x^2y^2 - 4y^2z^2$.
11. $m^4 - n^4$.
12. $a^8 - b^8$.
13. $m^{2n} - n^{2m}$.
14. $9a^{2n} - 4b^{4n}$.
15. $a^{16} - b^8$.

CASE V.

108. To resolve a quadratic trinomial into unequal factors.

1. What is the product of $x+2$ multiplied by $x+3$? What is the first term of the product? What is the last term? Of what two numbers is it the product? What is the coëfficient of the other term? How does it compare in value with 3 and 2?

2. What is the product of $x+3$ multiplied by $x+4$? What is the first term? Of what numbers is the last term the product? How does the coëfficient of the second term compare with 3 and 4?

3. What is the product of $x-10$ multiplied by $x-2$? How is each term of the product obtained from the quantities multiplied?

4. What is the product of $x+2$ multiplied by $x-5$? How is each term of the product obtained from the quantities multiplied?

109. A trinomial of the form of $x^2 \pm ax \pm b$, in which b is the product of two quantities and a their algebraic sum, is called a **Quadratic Trinomial**.

1. Resolve $x^2-9x-36$ into two factors.

PROCESS.

$$x^2-9x-36$$

$$36=\begin{cases}6\times 6\\4\times 9\\3\times 12\end{cases}$$

$$-9=3-12$$

$$(x+3)(x-12)$$

EXPLANATION.—The first term is evidently x. Since 36 is the product of the two other quantities, 6 and 6, or 4 and 9, or 3 and 12 are the other quantities.

Since their sum is -9, the quantities must be 3 and -12, for the other sets of factors of 36 can not be combined so as to produce this result.

Therefore, $(x+3)$ and $(x-12)$ are the factors.

RULE.—*For the first term of each factor take the square root of one term of the trinomial, and for the second term such quantities that their product will be another term of the trinomial, and their sum multiplied by the first term of the factor will be equal to the remaining term of the trinomial.*

Separate into factors the following trinomials:

2. x^2+3x+2.
3. $x^2+7x+12$.
4. $x^2-4x-21$.
5. $x^2-7x-18$.
6. x^2+6x+8.
7. $x^2+12x+32$.
8. $x^2-10x-39$.
9. $x^2-12x-64$.
10. $4x^2-10x+6$.
11. $9x^2-27x+18$.
12. $4x^2+16ax+12a^2$.
13. $9a^2+30ab+24b^2$.

CASE VI.

110. To resolve the difference of the same powers of two quantities into factors.

By performing the operations indicated in the following examples, it is found that,

1. $(a^2-b^2)\div(a-b)=a+b$.
2. $(a^3-b^3)\div(a-b)=a^2+ab+b^2$.

3. $(a^4 - b^4) \div (a - b) = a^3 + a^2b + ab^2 + b^3$.
4. $(a^5 - b^5) \div (a - b) = a^4 + a^3b + a^2b^2 + ab^3 + b^4$.

5. What will be the quotient when $a^6 - b^6$ is divided by $a - b$?

6. What will be the quotient when $a^7 - b^7$ is divided by $a - b$?

7. How does the first term of the quotient compare with the first term of the dividend? What quantities does the second term of the quotient contain? The third term? The fourth term?

8. What is the sign of each term?

9. What are the exponents of x and y, when the difference of the same powers of two quantities is divided by the difference of the quantities?

111. Principle.—*The difference of the same powers of two quantities is always divisible by the difference of the quantities.*

1. Write out the quotient of $(x^8 - y^8) \div (x - y)$.
2. Write out the quotient of $(x^9 - y^9) \div (x - y)$.
3. Write out the quotient of $(x^4 - 1) \div (x - 1)$.
4. Write out the quotient of $(x^4 - 16) \div (x - 2)$.
5. Write out the quotient of $(x^8 - y^8) \div (x^2 - y^2)$.

112. A course of reasoning which discloses the truth or falsity of a statement is a **Demonstration.**

113. The following is a general demonstration of the principle given in Art. 111:

Let x and y represent any two quantities, and n the exponent of any power. Then, $x^n - y^n$ will be the difference of the same powers of two quantities, and $x - y$ the difference of the two quantities.

PROCESS.

$$
\begin{array}{lll}
 & x^n - y^n & \underline{\left|\, x - y \right.} \\
 & \underline{x^n - x^{n-1}y} & x^{n-1} + x^{n-2}y \\
\text{1st Rem.,} & \quad x^{n-1}y - y^n & \\
 & \quad \underline{x^{n-1}y - x^{n-2}y^2} & \\
\text{2d Rem.,} & \qquad x^{n-2}y^2 - y^n & \\
n\text{th Rem.,} & \qquad x^{n-n}y^n - y^n & \\
 & \qquad x^0y^n - y^n & \\
 & \qquad y^n - y^n = 0 &
\end{array}
$$

DEMONSTRATION.—By dividing $x^n - y^n$ until several remainders are obtained, it is found that the first term of the first remainder is $x^{n-1}y$; of the second, $x^{n-2}y^2$; of the third, $x^{n-3}y^3$; of the fourth, $x^{n-4}y^4$, and, consequently, of the nth $x^{n-n}y^n$. But x^{n-n} is x^0, which equals 1 (Art. 93, 1). Therefore, the first term of the nth remainder reduces to y^n.

Since the second term in the nth remainder is $-y^n$, the entire nth remainder is $y^n - y^n$, or 0; that is, there is no remainder, and the division is exact. Therefore, $x^n - y^n$ is divisible by $x - y$ when x and y represent any two quantities and n the exponent of any power; or,

The difference of the same powers of two quantities is always divisible by the difference of the quantities.

114. By performing the operations indicated in the following examples, it is found that,

1. $(x^2 - y^2) \div (x + y) = x - y$.
2. $(x^3 - y^3) \div (x + y) = x^2 - xy + y^2$. Rem. $-2y^3$.
3. $(x^4 - y^4) \div (x + y) = x^3 - x^2y + xy^2 - y^3$.
4. $(x^5 - y^5) \div (x + y) = x^4 - x^3y + x^2y^2 - xy^3 + y^4$. Rem. $-2y^5$.

5. What is the quotient of $x^5 - y^5$ divided by $x + y$?

6. What are the signs of the terms of the quotient when the difference of the same powers is divided by the sum of the quantities? What is the law of the exponents in the quotient?

7. What are the exponents of x and y when the difference of the same powers of two quantities is exactly divisible by the sum of the quantities?

115. Principle.—*The difference of the same powers of two quantities is divisible by the sum of the quantities only when the exponents are even.*

1. Write out the quotient of $(x^8 - y^8) \div (x + y)$.
2. Write out the quotient of $(x^{10} - y^{10}) \div (x + y)$.
3. Write out the quotient of $(x^4 - 1) \div (x + 1)$.
4. Write out the quotient of $(x^4 - 16) \div (x + 2)$.
5. Write out the quotient of $(x^8 - y^8) \div (x^2 + y^2)$.

116. The following is a general demonstration of the principle in Art. 115:

PROCESS.

$$
\begin{array}{l|l}
 & x^n - y^n \quad \big|\; x + y \\
 & \underline{x^n + x^{n-1}y} \;\big|\; \underline{x^{n-1} - x^{n-2}y + x^{n-3}y^2 - x^{n-4}y^3} \\
\text{1st Rem.,} & -x^{n-1}y - y^n \\
 & \underline{-x^{n-1}y - x^{n-2}y^2} \\
\text{2d Rem.,} & \quad x^{n-2}y^2 - y^n \\
 & \quad \underline{x^{n-2}y^2 + x^{n-3}y^3} \\
\text{3d Rem.,} & \qquad -x^{n-3}y^3 - y^n \\
 & \qquad \underline{-x^{n-3}y^3 - x^{n-4}y^4} \\
\text{4th Rem.,} & \qquad\quad x^{n-4}y^4 - y^n
\end{array}
$$

Demonstration.—By dividing $x^n - y^n$ until several remainders are obtained, it is found that the first term of the first remainder is $-x^{n-1}y$; of the third, $-x^{n-3}y^3$; of the fifth, $-x^{n-5}y^5$; and of the nth remainder, when n is an odd number, $-x^{n-n}y^n$. But $-x^{n-n}$ is $-x^0$, which is equal to -1. Therefore, the first term of the nth remainder, when n is odd, reduces to $-y^n$. Since the second term of the nth remainder is $-y^n$, the entire nth remainder, when n is an odd number, is $-y^n - y^n$, or $-2y^n$. Therefore, $x^n - y^n$ is not exactly divisible by $x + y$ when n is an odd number. Hence, the

difference of the same odd powers of two quantities is not divisible by the sum of the quantities.

The first term of the second remainder is $x^{n-2}y^2$; of the fourth, $x^{n-4}y^4$; of the sixth, $x^{n-6}y^6$; and of the nth remainder, when n is an even number, $x^{n-n}y^n$. But $x^{n-n}y^n$ is equal to x^0y^n, or y^n. And since the second term of the nth remainder is $-y^n$, the entire nth remainder, when n is an even number, is $y^n - y^n$, or 0. Hence, $x^n - y^n$ is exactly divisible by $x + y$ when n is an even number; or,

The difference of the same powers of two quantities is divisible by the sum of the quantities only when the exponents are even.

CASE VII.

117. To resolve the sum of the same powers of two quantities into factors.

By performing the operations indicated in the following examples, it is found that,

1. $(x^2 + y^2) \div (x + y) = x - y$. Rem. $2y^2$.
2. $(x^3 + y^3) \div (x + y) = x^2 - xy + y^2$.
3. $(x^4 + y^4) \div (x + y) = x^3 - x^2y + xy^2 - y^3$. Rem. $2y^4$.
4. $(x^5 + y^5) \div (x + y) = x^4 - x^3y + x^2y^2 - xy^3 + y^4$.

5. What are the signs of the terms of the quotient? What is the law of the exponents?

6. What are the exponents of x and y when the sum of the same powers of two quantities is exactly divisible by the sum of the quantities?

118. PRINCIPLE.—*The sum of the same powers of two quantities is divisible by the sum of the quantities only when the exponents are odd.*

1. Write out the quotient of $(x^7 + y^7) \div (x + y)$.
2. Write out the quotient of $(x^9 + y^9) \div (x + y)$.
3. Write out the quotient of $(x^5 + 1) \div (x + 1)$.

119. The following is a general demonstration of the principle in Art. 118:

PROCESS.

$$
\begin{array}{ll|l}
 & x^n + y^n & x + y \\
 & \underline{x^n + x^{n-1}y} & \overline{x^{n-1} - x^{n-2}y + x^{n-3}y^2 - x^{n-4}y^3} \\
\text{1st Rem.,} & -x^{n-1}y + y^n & \\
 & \underline{-x^{n-1}y - x^{n-2}y^2} & \\
\text{2d Rem.,} & \qquad x^{n-2}y^2 + y^n & \\
 & \qquad \underline{x^{n-2}y^2 + x^{n-3}y^3} & \\
\text{3d Rem.,} & \qquad\qquad -x^{n-3}y^3 + y^n & \\
 & \qquad\qquad \underline{-x^{n-3}y^3 - x^{n-4}y^4} & \\
\text{4th Rem.,} & \qquad\qquad\qquad x^{n-4}y^4 + y^n &
\end{array}
$$

DEMONSTRATION.—Dividing and reasoning, as in the previous demonstrations, the nth remainder, when n is even, reduces to $y^n + y^n$, or $2y^n$. Hence, $x^n + y^n$ is not exactly divisible by $x + y$ when n is even, or the sum of the same powers of two quantities is not exactly divisible by the sum of the quantities when the exponents are even. The first term of the nth remainder, when n is odd, reduces to $-y^n$, and the entire remainder is $-y^n + y^n$, or 0. Hence, $x^n + y^n$ is exactly divisible by $x + y$ when n is an odd number; or,

The sum of the same powers of two quantities is divisible by the sum of the quantities only when the exponents are odd.

120. Perform the operations indicated in the following examples, and write down the quotients and remainders:

1. $(x^2 + y^2) \div (x - y)$.
2. $(x^3 + y^3) \div (x - y)$.
3. $(x^4 + y^4) \div (x - y)$.
4. $(x^5 + y^5) \div (x - y)$.

5. From the results, discover whether the sum of the same powers of two quantities is divisible by the difference of the quantities.

6. Demonstrate the truth of the following principle:

121. PRINCIPLE.—*The sum of the same powers of two quantities is never divisible by the difference of the quantities.*

COMMON DIVISORS.

122. 1. What number will exactly divide both 15 and 20?

2. What quantity will exactly divide both $3a$ and $2a$?

3. What quantity will exactly divide both $3a^2$ and $2ax$?

4. Give all the exact divisors of $12a^2xy$ and $18ax^2y$. What is the greatest, or highest, of these common divisors?

5. What is the greatest, or highest, common divisor of $24a^2b^2c$ and $48a^2bc^2$?

6. What prime factors, or divisors, are common to $24a^2b^2c$ and $48a^2bc^2$?

7. How may the greatest, or highest, common divisor of $24a^2b^2c$ and $48a^2bc^2$ be obtained from these factors?

8. How may the greatest, or highest, common divisor of $15x^2y^3z$ and $20x^2y^2z$ be obtained from their prime factors?

DEFINITIONS.

123. A **Common Divisor** of two or more quantities is an exact divisor of each of them.

Thus, $6a$ is a common divisor of $12a$, $24a^2c$, and $30a^2y$.

124. The **Greatest, or Highest,* Common Divisor** of two or more quantities is the greatest or highest quantity that is an exact divisor of each of them.

Thus, $4a^2x$ is the greatest, or highest, common divisor of $12a^3xy$, $8a^2x^2y$, and $4a^2xz$.

---

* Strictly speaking, *Highest Common Divisor* would be the appropriate term to apply to literal quantities, because, although x^2 is a *higher power* than x, the *value* of x^2 may be *less* than the value of x, but common usage is followed in employing *Greatest Common Divisor* to include both.

125. When quantities have no common divisor they are prime to each other.

Thus, $5x$, $3y$, and $8z$ are prime to each other.

126. PRINCIPLE.—*The greatest common divisor of two or more quantities is the product of all their common prime factors.*

CASE I.

127. To find the greatest common divisor of quantities that can be factored readily.

1. What is the greatest common divisor of $8a^2b^2c^3$ and $12ab^2c^2$?

PROCESS.

$$\begin{array}{l}
8a^2b^2c^3 = 2\times 2\times 2\times a\times a\times b\times b\times c\times c\times c \\
12ab^2c^2 \;= 3\times 2\times 2\times a\times b\times b\times c\times c \\
\hline
\text{G. C. D.} = 2\times 2\times a\times b\times b\times c\times c = 4ab^2c^2
\end{array}$$

EXPLANATION.—Since the greatest common divisor is the product of all the common prime factors (Prin.), the quantities are separated into their prime factors. The only prime factors common to the given quantities are 2, 2, a, b, b, c, c; and their product, $4ab^2c^2$, is therefore the greatest common divisor.

2. What is the greatest common divisor of $a(x^2-y^2)$ and $a(x^2+2xy+y^2)$?

PROCESS.

$$\begin{array}{ll}
a(x^2-y^2) & = a(x+y)\ (x-y) \\
a(x^2+2xy+y^2) & = a(x+y)\ (x+y) \\
\hline
\text{G. C. D.} & = a\times(x+y) = a(x+y)
\end{array}$$

EXPLANATION.—Reasoning as in the preceding example, the quantities are separated into their prime factors, and the product of the common factors will be the greatest common divisor.

The common factors are a and $(x+y)$; therefore. $a(x+y)$ is the greatest common divisor.

RULE.—*Separate the quantities into their prime factors, and find the product of all the common factors.*

Find the greatest common divisor of the following:

3. $12m^2n^3x^2$ and $18m^2nx^2$.
4. $16r^4s^3x^2$ and $20r^2s^2x^2$.
5. $21x^4y^2z^3$ and $14x^2y^3z^3$.
6. $15x^5y^2z^5$ and $20x^5yz^2$.
7. $11a^2xy$, $8ax^2y$, and $9axy$.
8. $15a^3x^2y^2$, $9a^2x^2y^3$, and $8a^2xy^2$.
9. $18b^2c^2d^3$, $8b^2c^2d^2$, and $12ab^2c$.
10. $10c^3x^3y^3$, $8a^2x^2y^3$, and $12a^2xy^2$.
11. $18r^2s^2t^2$, $10r^2s^3t$, and $16r^2s^2t^2$.
12. $20a^3x^3y^3$, $15a^2x^2y^3$, and $10a^2xy^2$.
13. $12x^3y^2z^2$, $18x^4y^3z^3$, and $15x^2y^4z^2$.
14. a^2-b^2 and $a^2-2ab+b^2$.
15. x^2-2x and $2xy^2-4y^2$.
16. $16x^2-y^2$ and $16x^2-8xy+y^2$.
17. $x^2-2x-15$ and $x^2+9x+18$.
18. $x^2+9x+20$ and $x^2+2x-15$.
19. x^2+x-30 and $x^2+12x+36$.
20. x^2-x-12 and $x^2-4x-21$.
21. $x^2+9x+14$ and $x^2+2x-35$.
22. x^2+x-30 and $x^2+9x+18$.
23. $a(x^4-y^4)$ and $x^4+2x^3y+x^2y^2$.

CASE II.

128. To find the greatest common divisor of polynomials.

1. What are the exact divisors of 10? What are they of 2 times 10 or 20? Of 3 times 10 or 30? Of any number of times 10?

2. What are the exact divisors of ax^2? What are these also of 2 times ax^2 or $2ax^2$? Of c times ax^2 or cax^2?

3. If a quantity is an exact divisor of some quantity, what will it also be of any number of times that quantity?

4. Since 15 and 20 are each divisible by 5, what must they each be of 5?

5. Since they are each some number of times 5, what will their sum be of 5? What will their difference be of 5?

6. Since a is a divisor of $2ab$ and $3ac$, what will it be also of their sum? What of their difference?

7. If a quantity is an exact divisor of each of two quantities, what is it of their sum? What of their difference?

8. What is the greatest common divisor of 10 and 15? Of 2 times 10, or 20, and 15? Of 4 times 10, or 40, and 15? Of $10 \div 2$, or 5 and 15?

9. What factors of these multipliers and divisors of 10 are factors of 15?

10. What is the greatest common divisor of 10 and 3 times 15, or 45? Of 10 and 7 times 15, or 105? Of 10 and $15 \div 3$ or 5?

11. What factors of these multipliers and divisors of 15 are factors of 10?

12. By what quantities, then, may either quantity be multiplied or divided without changing their greatest common divisor?

129. Principles.—1. *A divisor of any quantity is a divisor of any number of times that quantity.*

2. *A divisor of two or more quantities is a divisor of their sum and of the difference between any two of them.*

3. *The greatest common divisor of two or more quantities is not affected by multiplying or dividing any of them by quantities which are not factors of the others.*

130. 1. What is the greatest common divisor of $6x^2 + 37x + 35$ and $3x^2 + 17x + 10$?

PROCESS.

$$\begin{array}{r|l}
6x^2 + 37x + 35 & \underline{3x^2 + 17x + 10} \\
\underline{6x^2 + 34x + 20} & \underline{2\qquad\qquad} \\
3\underline{)3x + 15} & \\
x + 5 &
\end{array}$$

$$\begin{array}{r|l}
3x^2 + 17x + 10 & \underline{x + 5} \\
\underline{3x^2 + 15x\qquad} & \underline{3x + 2} \\
2x + 10 & \\
\underline{2x + 10} &
\end{array}$$

EXPLANATION.—The greatest common divisor of two quantities can not be greater than the smaller quantity; therefore, the greatest common divisor of these two quantities can not be greater than $3x^2 + 17x + 10$. $3x^2 + 17x + 10$ will be the greatest common divisor if it is exactly contained in $6x^2 + 37x + 35$. By trial, it is found that it is not an exact divisor of $6x^2 + 37x + 35$, since there is a remainder of $3x + 15$. Therefore, $3x^2 + 17x + 10$ is not the greatest common divisor.

Since $6x^2 + 37x + 35$ and $6x^2 + 34x + 20$, which is 2 times $3x^2 + 17x + 10$, are each divisible by the greatest common divisor, their difference, $3x + 15$, must contain the greatest common divisor (Prin. 2). Therefore, the greatest common divisor can not be greater than $3x + 15$.

Since 3 is a factor of $3x + 15$, but not of the quantity whose greatest common divisor is sought, $3x + 15$ may be divided by 3 without changing the greatest common divisor (Prin. 3). Therefore, the greatest common divisor can not be greater than $x + 5$.

$x + 5$ will be the greatest common divisor if it is exactly contained in $3x^2 + 17x + 10$, since if it is contained in $3x^2 + 17x + 10$ it will be contained in *twice* $3x^2 + 17x + 10$, or $6x^2 + 34x + 20$, and in the sum of $3x + 15$ and $6x^2 + 34x + 20$, or $6x^2 + 37x + 35$. By trial it is found that $x + 5$ is an exact divisor of $3x^2 + 17x + 10$.

Therefore, $x + 5$ is the greatest common divisor.

2. What is the greatest common divisor of $3x^2 + 11x + 6$ and $2x^2 + 11x + 15$?

PROCESS.

$$
\begin{array}{r|l}
3x^2 + 11x + 6 & \\
2 & \\
\hline
6x^2 + 22x + 12 & 2x^2 + 11x + 15 \\
6x^2 + 33x + 45 & 3 \\
\hline
-11)\,-11x - 33 &
\end{array}
$$

$$
\begin{array}{r|l}
2x^2 + 11x + 15 & x + 3 \\
2x^2 + 6x & 2x + 5 \\
\hline
5x + 15 & \\
5x + 15 & \\
\hline
\end{array}
$$

EXPLANATION.—If $3x^2 + 11x + 6$ is divided by $2x^2 + 11x + 15$, the quotient will be a fraction. To avoid the fractional quotient, we multiply $3x^2 + 11x + 6$ by 2 without changing the greatest common divisor, since 2 is not a factor of the quantities whose greatest common divisor is sought. (Prin. 3.)

If the preceding divisor, $2x^2 + 11x + 15$, is divided by $-11x - 33$, the quotient will be a fraction. This result may be avoided by dividing $-11x - 33$ by the factor -11 without changing the greatest common divisor, since it is not a factor of the quantities whose greatest common divisor is sought. (Prin. 3.)

Dividing by $x + 3$, the division is exact.

Therefore, $x + 3$ is the greatest common divisor.

RULE.—*Divide the greater quantity by the less, and if there be a remainder, divide the less quantity by it, then the preceding divisor by the last remainder, and so on until nothing remains. The last divisor will be the greatest common divisor.*

If more than two quantities are given, find the greatest common divisor of any two, then of this divisor and another, and so on. The last divisor will be the greatest common divisor.

1. If any quantity contains a factor not found in the other, the factor may be omitted before beginning the process.

2. When necessary, the dividend may be multiplied by any quantity not a factor of the divisor.

3. The signs of all the terms of either dividend or divisor, or both, may be changed without changing the greatest common divisor.

Find the greatest common divisor of the following:

3. $2x^2 - 16x + 14$ and $x^2 - 5x - 14$.

4. $3x^2 + 14x + 8$ and $4x^2 + 19x + 12$.

5. $6x^2 - 23x + 15$ and $2x^2 - 12x + 18$.

6. $4x^2 + 21x - 18$ and $2x^2 + 15x + 18$.

7. $21x^2 - 26x + 8$ and $6x^2 - x - 2$.

8. $x^2 - 6xy + 8y^2$ and $x^2 - 8xy + 16y^2$.

9. $x^3 - y^3$ and $x^2 - 2xy + y^2$.

10. $x^4 - 2x^2 + 1$ and $x^4 - 4x^3 + 6x^2 - 4x + 1$.

11. $2x^3 + 6x^2 + 6x + 2$ and $6x^3 + 6x^2 - 6x - 6$.

12. $3x^3 + 3x^2 - 15x + 9$ and $3x^4 + 3x^3 - 21x^2 - 9x$.

13. $20x^4 + x^2 - 1$ and $25x^4 + 5x^3 - x - 1$.

14. $x^2 - 9$, $x^2 - 3x - 18$, and $x^2 + 11x + 24$.

15. $x^2 - 3x - 28$, $x^2 - 11x + 28$, and $x^2 - 15x + 56$.

16. $x^2 + 6x + 9$, $x^3 - x^2 - 12x$, and $x^2 - 4x - 21$.

17. $a^4 - b^4$, $a^3 + a^2b - ab^2 - b^3$, and $a^4 - 2a^2b^2 + b^4$.

18. $x^4 + 5x^3 + 6x^2$, $x^3 + 3x^2 + 3x + 2$, and $3x^3 + 8x^2 + 5x + 2$.

19. $a^3 + 3a^2b + 3ab^2 + b^3$, $4a^2b^2 + 12ab^3 + 8b^4$, and $a^2 - b^2$.

20. $9x^4 + 12x^3 + 10x^2 + 4x + 1$ and $3x^4 + 8x^3 + 14x^2 + 8x + 3$.

21. $x^4 + 3x^3 + 9x^2 + 12x + 20$ and $x^5 + 6x^3 + 6x^2 + 8x + 24$.

22. $3a^2x^2 + a^2x + 2a^2 + 12x^2 + 4x + 8$ and $a^2x^2 + 3a^2x + 4a^2 + 4x^2 + 12x + 16$.

COMMON MULTIPLES.

131. 1. What is the least number that will contain 10 and 15?

2. What prime factors are common to 10 and 15? What factor occurs in 10 that does not in 15? What factor in 15 is not found in 10? What are all the factors of 15 and those in 10 not found in 15? What is their product?

3. What quantity will exactly contain 2, 3, a and b? What will each of them be of their product?

4. What is the least or lowest quantity that will exactly contain $3a$ and $4ab$?

5. What factor of $3a$ is not found in $4ab$? What is the product of 3 multiplied by $4ab$?

6. To what, then, is the least or lowest common multiple of several quantities equal?

DEFINITIONS.

132. A **Multiple** of a quantity is a quantity that will exactly contain it.

Thus, a^2x is a multiple of a, a^2, and x.

133. A **Common Multiple** of two or more quantities is a quantity that will exactly contain each of them.

Thus, $4b^2c$ is a common multiple of $2b$ and c.

134. The **Least, or Lowest,* Common Multiple** of two or more quantities is the least or lowest quantity that will exactly contain each of them.

Thus, $2bc$ is the least common multiple of $2b$ and c.

---

*Common usage is followed in employing the term *Least Common Multiple*, although *Lowest Common Multiple* would be the appropriate term to apply to literal quantities.

135. PRINCIPLE.—*The least common multiple of two or more quantities is equal to the product of the highest quantity multiplied by all the factors of the other quantities not contained in the highest quantity.*

136. 1. What is the least common multiple of $3x^2y^2zv$ and $5x^2y^3z^2$?

PROCESS.

$$\begin{aligned} 3x^2y^2zv &= 3 \times x^2 \times y^2 \times z \times v \\ 5x^2y^3z^2 &= 5 \times x^2 \times y^3 \times z^2 \end{aligned}$$
$$\text{L. C. M.} = 5 \times 3 \times x^2 \times y^3 \times z^2 \times v = 15x^2y^3z^2v$$

EXPLANATION.—Since the least common multiple is equal to the product of the highest quantity multiplied by the factors of the other quantity not found in the highest quantity (Prin.), for convenience in determining what factors of the other quantity are not found in the higher, the quantities are separated into their prime factors. Thus, the factors of the least common multiple are seen to be 5, 3, x^2, y^3, z^2, and v.

Hence, their product, $15x^2y^3z^2v$, is their least common multiple.

2. What is the least common multiple of $a^2 - a - 12$ and $a^2 - 4a - 21$?

PROCESS.

$$\frac{(a^2 - a - 12)(a^2 - 4a - 21)}{a + 3} =$$
$$(a - 4)(a^2 - 4a - 21) =$$
$$a^3 - 8a^2 - 5a + 84$$

EXPLANATION.—Since the product of any two quantities is their common multiple, it follows that if their common factors are omitted from the product, the result will be the *least* common multiple. Since their common factors or divisors will be the greatest common divisor of the quantities, the product of the two quantities divided by their greatest common divisor will be their least common multiple.

Their greatest common divisor is $a + 3$; omitting this factor from dividend and divisor, the result is $(a - 4)(a^2 - 4a - 12)$, which is equal to $a^3 - 8a^2 - 5a + 48$, their least common multiple.

RULE.—*Separate the quantities into their prime factors. Multiply the factors of the highest quantity by the factors of the other quantities not found in it; or, divide the product of the quantities by their greatest common divisor.*

Find the least common multiple of the following:

3. $8a^2b^2c^3$ and $10a^2bc$.
4. $10x^2y^2z$, $20x^3y^2z$, and $25x^2y^3z^3$.
5. $14a^2b^2c^2$, $7b^2x^2y$, and $35abcx$.
6. $12m^2n^2y^2$, $18mny^3$, and $24m^2n^3y$.
7. $18r^2s^2z^3$, $9r^3sz^2$, and $36rs^3z^4$.
8. $x^2 - y^2$ and $x^2 - 2xy + y^2$.
9. $x^2 - y^2$ and $x^2 + 2xy + y^2$.
10. $x^2 - y^2$, $x^2 - 2xy + y^2$, and $x^2 + 2xy + y^2$.
11. $x^2 - y^2$ and $x^3 - y^3$.
12. $a^2(x - z)$ and $y^2(x^2 - z^2)$.
13. $x^2 - 1$, $x^2 + 1$, and $x^4 - 1$.
14. $2x(x - y)$, $4xy(x^2 - y^2)$, and $6xy^2(x + y)$.
15. $x^2 - x$, $x^3 - 1$, and $x^3 + 1$.
16. $x^2 - 1$, $x^2 - x$, and $x^3 - 1$.
17. $4(1 + x)$, $4(1 - x)$, and $8(1 - x^2)$.
18. $x^2 + 5x + 6$ and $x^2 + 6x + 8$.
19. $a^2 - a - 20$ and $a^2 + a - 12$.
20. $x^2 - 9x - 22$ and $x^2 - 13x + 22$.
21. $x^2 - 8x + 15$ and $x^2 + 2x - 5$.
22. $x^3 + x^2y + xy^2 + y^3$ and $x^3 - x^2y + xy^2 - y^3$.
23. $x^3 - x^2y + xy^2 - y^3$ and $x^3 + x^2y - xy^2 - y^3$.
24. $a^3 - 2a^2 + 4a - 8$ and $a^3 + 2a^2 - 4a - 8$.
25. $x^2 + y^2$, $x^3 - xy^2$, and $x^3 + xy^2 + x^2y + y^3$.
26. $x^2 - 4$, $x^2 - x - 6$, and $x^3 - 3x^2 - 4x + 12$.
27. $x - 5$, $x^2 - 2ax + a^2$, $x^2 - 10x + 25$, and $x^2 + 5a - 5x - ax$.
28. $x^4 - 16$, $x^2 + 4x + 4$, and $x^2 - 4$.

FRACTIONS.

137. 1. When any thing is divided into two equal parts, what is one part called? How is it expressed? What does $\frac{1}{3}$ represent? $\frac{2}{3}$? $\frac{3}{5}$?

2. What does $\dfrac{a}{2}$ represent? $\dfrac{2a}{5}$? $\dfrac{2a}{7}$? $\dfrac{3a}{9}$?

3. How may one fifth of x be expressed? Two-thirds of b? Three-sevenths of y? Eight-elevenths of z?

DEFINITIONS.

138. A **Fraction** is one or more of the equal parts of a unit.

139. The **Unit of a Fraction** is the unit which is divided into equal parts.

140. A **Fractional Unit** is one of the equal parts into which a unit is divided.

141. Since a fraction is one or more of the equal parts of any thing, to express a fraction two numbers, or quantities, are necessary, one to express the number of equal parts into which the unit has been divided; the other to express how many parts form the fraction. These numbers, or quantities, are written one above the other, with a horizontal line between them.

142. The **Denominator** is the number, or quantity, which shows into how many equal parts the unit is divided.

It is written below the line.

Thus, in the fraction $\frac{a}{b}$, b is the denominator. It shows that the unit of the fraction has been divided into b equal parts.

143. The **Numerator** is the number, or quantity, which shows how many fractional units form the fraction.

It is written above the line.

Thus, in the fraction $\frac{a}{b}$, a is the numerator. It shows how many fractional units form the fraction.

144. The numerator and denominator are called the **Terms of a Fraction.**

145. An indicated process in division may be written in the *form of a fraction*, the numerator being the dividend and the denominator the divisor.

146. A quantity, no part of which is in the form of a fraction, is called an **Entire Quantity.**

Thus, $2a$, $3c$, $2x + y$, etc., are entire quantities.

147. A **Mixed Quantity** is a quantity composed of an entire quantity and a fraction.

Thus, $2a + \frac{3b}{7}$, $2x + 2y + \frac{3z + 2}{x + 7}$, are mixed quantities.

148. The **Sign of a Fraction** is the sign written before the dividing line. This sign belongs to the fraction as a *whole*, and not to either the numerator or denominator.

Thus, in $-\frac{x + y}{2z}$ the sign of the fraction is $-$, while the signs of the quantities x, y, and $2z$ are $+$. The sign before the dividing line shows whether the fraction is to be added or subtracted.

REDUCTION OF FRACTIONS.

CASE I.

149. To reduce fractions to higher or lower terms.

1. How many fourths are there in one-half? How many eighths?

2. How many sixths are there in one-third? How many ninths? How are the terms of the fraction $\frac{2}{6}$ obtained from $\frac{1}{3}$? $\frac{3}{9}$ from $\frac{1}{3}$?

3. How many fourths are there in $\frac{b}{2}$? $\frac{x}{2}$? $\frac{y}{2}$?

4. How many sixths are there in $\frac{b}{3}$? How many ninths? How are the terms of the fraction $\frac{2b}{6}$ obtained from its equivalent $\frac{b}{3}$? $\frac{3b}{9}$ from $\frac{b}{3}$?

5. What, then, may be done to the terms of a fraction without changing the value of the fraction?

6. How many fourths are there in $\frac{2}{8}$? In $\frac{4}{8}$? In $\frac{6}{8}$?

7. How many thirds are there in $\frac{2}{6}$? In $\frac{3}{9}$? In $\frac{4}{12}$? How are the terms of the fraction $\frac{1}{3}$ obtained from $\frac{2}{6}$? From $\frac{3}{9}$? From $\frac{4}{12}$?

8. How many thirds are there in $\frac{2b}{6}$? In $\frac{3b}{9}$? In $\frac{4b}{12}$? How are the terms of the equivalent fraction $\frac{b}{3}$ obtained from these fractions?

9. What else may be done to the terms of a fraction, besides multiplying them by the same quantity, that will not change the value of the fraction?

150. Reduction of Fractions is the process of changing their form without changing their value.

A fraction is expressed in its **Lowest Terms** when its numerator and denominator have no *common divisor.*

151. PRINCIPLE.—*Multiplying or dividing both terms of a fraction by the same quantity does not change the value of the fraction.*

EXAMPLES.

1. Change $\frac{3}{2b}$ to a fraction whose denominator is $6b^2$.

PROCESS.

$$\frac{3}{2b}$$

$$6b^2 \div 2b = 3b$$

$$\frac{3 \times 3b}{2b \times 3b} = \frac{9b}{6b^2}$$

EXPLANATION.—Since the fraction is to be changed to an equivalent fraction expressed in higher terms, the terms of the fraction must be multiplied by the same quantity, so that the value of the fraction may not be changed (Prin.). In order to produce the required denominator, the given denominator must be multiplied by $3b$; consequently, the numerator must be multiplied by $3b$ also.

2. Reduce $\frac{15x^4y^2}{25x^5y}$ to its lowest terms.

PROCESS.

$$\frac{15x^4y^2}{25x^5y} = \frac{3y}{5x}$$

EXPLANATION.—Since the fraction is to be changed to an equivalent fraction expressed in its lowest terms, the terms of the fraction may be divided by any quantity that will exactly divide them (Prin.). Dividing by the factors 5, x^4, and y, the expression is reduced to its lowest terms, for the terms are prime to each other; or,

The terms may be divided by their greatest common divisor.

152. To express a fraction in higher terms.

RULE.—*Multiply the terms of the fraction by such a quantity as will change the given term to the required term.*

153. To express a fraction in its lowest terms.

RULE.—*Divide the terms of the fraction by any common divisor, and continue to divide thus until they have no common divisor*; or,

Divide the terms of the fraction by their greatest common divisor.

3. Change $\frac{3a}{7}$ to a fraction whose denominator is 28.

4. Change $\frac{5x^2}{6}$ to a fraction whose denominator is 36.

5. Change $\frac{2a+4b}{3}$ to a fraction whose denominator is 15.

6. Change $\frac{3x+7}{6}$ to a fraction whose denominator is 30.

7. Change $\frac{2x}{6x+3}$ to a fraction whose numerator is $6x$.

8. Change $\frac{3x}{6x-8}$ to a fraction whose numerator is $9x$.

9. Change $\frac{2ax}{3+2y}$ to a fraction whose numerator is $4ax^2$.

10. Change $\frac{2a+x}{a+b}$ to a fraction whose denominator is a^2-b^2.

11. Change $\frac{3x-y}{a+b}$ to a fraction whose denominator is $a^2+2ab+b^2$.

Reduce the following to their lowest terms:

12. $\frac{15x^2y^2z}{75xy^2z}$.

13. $\frac{21x^2y^2z^2}{28x^2y^3z^2}$.

14. $\frac{10abx^2y}{25abx^3y^2}$

15. $\frac{16xyz^3m}{24x^2y^2zm^3}$.

16. $\frac{21m^2n^2z^3}{12m^2z^4}$.

17. $\frac{24x^3y^2z^5}{12x^3y^3z^4}$.

18. $\frac{35x^5y^4z^7}{49x^4y^5z^8}$.

19. $\frac{22a^5x^2yz^4}{33a^4x^3y^2z^2}$.

20. $\frac{a^2-b^2}{a^2-2ab+b^2}$.

21. $\frac{a^2-b^2}{a^2+2ab+b^2}$.

22. $\frac{x^2-1}{x^2+2x+1}$.

23. $\frac{x^2-1}{2xy+2y}$.

24. $\frac{x^3-a^2x}{x^2-2ax+a^2}$.

25. $\frac{x^5-x^3y^2}{x^4-y^4}$.

26. $\frac{x^2+6x+9}{x^3-x^2-12x}$.

27. $\frac{x^2-3x-28}{x^2-11x+28}$.

CASE II.

154. To reduce an entire or mixed quantity to a fraction.

1. How many fifths are there in 3? In 4? In 10? In a? In x?

2. How many sevenths are there in 2? In 4? In 6? In b^2? In y^2?

3. How many fourths are there in $2\frac{1}{4}$? In $3\frac{3}{4}$? In $a+\frac{a}{4}$?

EXAMPLES.

1. Reduce $a+\frac{b}{c}$ to a fractional form.

PROCESS.

$$a=\frac{ac}{c}$$

$$a+\frac{b}{c}=\frac{ac}{c}+\frac{b}{c}=\frac{ac+b}{c}$$

EXPLANATION.—Since 1 is equal to $\frac{c}{c}$, a is equal to $\frac{ac}{c}$; consequently, $a+\frac{b}{c}=\frac{ac}{c}+\frac{b}{c}$, or $\frac{ac+b}{c}$.

RULE.—*Multiply the entire part by the denominator of the fraction; to this product add the numerator when the sign of the fraction is plus, and subtract it when it is minus, and write the result over the denominator.*

If the *sign of the fraction* is —, all the signs of the numerator must be changed when it is subtracted.

Reduce the following to fractional forms:

2. $2x + \frac{4y}{5}$.

3. $5x - \frac{3y}{4}$.

4. $4x - \frac{6z}{2}$.

5. $x + \frac{4y+3}{4}$.

6. $2a + \frac{3x+4}{4}$.

7. $2x + \frac{3y-4}{8}$.

8. $3x - \frac{2y+3}{6}$.

9. $5a - \frac{3x+4}{2}$.

10. $6a - \frac{3y+7}{4}$.

11. $3c + \frac{4a+b}{d}$.

12. $4a + \frac{3c-d}{cd}$.

13. $3x + \frac{6a-x}{ax}$.

14. $x + 4 + \frac{2c-d}{5}$.

15. $a - \frac{2ac-c^2}{a}$.

16. $2x - 5 - \frac{x^2+4}{x-2}$.

17. $a + x + \frac{a^2+x^2}{a-x}$.

18. $a + c + \frac{2ac-c^2}{a-c}$.

19. $x - y + \frac{x^2-y^2}{x+y}$.

20. $x + 4 - \frac{x^2-2}{x-4}$.

21. $a + x - \frac{4ax-5x^2}{a-x}$.

22. $a - b - \frac{a^2+b^2}{a-b}$.

23. $m + n - \frac{2mn+n^2}{m-n}$.

CASE III.

155. To reduce a fraction to an entire or a mixed quantity.

1. How many units are there in $\frac{16}{4}$? In $\frac{15}{4}$? In $\frac{25}{8}$?
2. How many units are there in $\frac{8}{8}$? In $\frac{25}{9}$? In $\frac{31}{7}$?
3. How many units in $\frac{6a+6b}{6}$? In $\frac{8x-4}{4}$? In $\frac{5m+n}{5}$?

EXAMPLES.

1. Reduce $\frac{bx+d}{b}$ to a mixed quantity.

PROCESS.

$$\frac{bx+d}{b} = x + \frac{d}{b}$$

EXPLANATION.—Since a fraction may be regarded as an expression of unexecuted division, by performing the division indicated, the fraction is changed into the form of a mixed quantity.

Reduce the following to entire or mixed quantities:

2. $\frac{a^2+c^2}{a}$.
3. $\frac{bx+cd}{b}$.
4. $\frac{2ab+b^2}{a+b}$.
5. $\frac{a^2-x^2}{a-x}$.
6. $\frac{a^2-x^2}{a+x}$.
7. $\frac{x^3+1}{x+1}$.
8. $\frac{x^3+1}{x-1}$.
9. $\frac{x^2+2ax+x^2}{a+x}$.
10. $\frac{a^3+b^3}{a-b}$.
11. $\frac{5ay+ax+x}{ax}$.
12. $\frac{2a^2-2b^2}{a+b}$.
13. $\frac{x^2+2xy+2y^2+x}{x+y}$.
14. $\frac{a^3-b^3}{a-b}$.
15. $\frac{x^2+xy+y^2}{x+y}$.

CASE IV.

156. To transfer a factor from one term of a fraction to the other.

1. To what is the reciprocal of any quantity equal?

2. To what is any quantity with a negative exponent equal?

3. Change $\frac{1}{x^2}$ to an equivalent expression which is not in the reciprocal form.

4. Change x^{-2} to the reciprocal form.

157. PRINCIPLE.—*Any quantity may be changed from one term of a fraction to the other by changing the sign of its exponent.*

EXAMPLES.

1. Change $\frac{a^2x^2}{b^2c^2}$ to an equivalent expression in the form of an entire quantity.

PROCESS.

$$\frac{a^2x^2}{b^2c^2}=a^2x^2 \times \frac{1}{b^2c^2}$$

$$\frac{1}{b^2c^2}=b^{-2}c^{-2}$$

$$a^2x^2 \times \frac{1}{b^2c^2}=a^2x^2 \times b^{-2}c^{-2}=a^2x^2b^{-2}c^{-2}$$

EXPLANATION.—Since $\frac{a^2x^2}{b^2c^2}$ is equal to $a^2x^2 \times \frac{1}{b^2c^2}$, and $\frac{1}{b^2c^2} = b^{-2}c^{-2}$ (Prin.), $\frac{a^2x^2}{b^2c^2}$ equals $a^2x^2 \times b^{-2}c^{-2}$, which is $a^2x^2b^{-2}c^{-2}$.

RULE.—*Change the factors from one term of the fraction to the other and change the signs of the exponents.*

Express the following in the form of entire quantities:

2. $\frac{a^4}{c^2x^3}$.

3. $\frac{x^2y}{c^2d}$.

4. $\frac{xy^2}{x^2y^2}$.

5. $\frac{ab^2}{ac^2}$.

6. $\frac{xyz^3}{x^2y^2z}$.

7. $\frac{a+x}{ax}$.

8. $\frac{cxy-z}{cxz}$.

9. $\frac{x^2-2xy+y^2}{x^2-y^2}$.

10. $\frac{a+x}{a^2-x^2}$.

11. $\frac{x^2-y^2}{x^2y^2}$.

Change the following to equivalent quantities having positive exponents:

12. $\frac{3x^{-2}}{x^{-4}y^{-2}}$.

13. $\frac{4ac^{-2}}{3a^{-1}}$.

14. $\frac{3axy}{4a^{-1}x^{-2}}$.

15. $\frac{(a^2-b^2)(a-c)^{-1}}{(a+c)(a^2+b^2)^{-1}}$.

16. $\frac{4(a-x)^{-2}}{a-x}$.

17. $\frac{(x^2+y^2)(x-y)^{-1}}{(x^2-y^2)^{-1}(x+y)}$.

18. $\frac{5(x-3)^{-2}}{(x+3)}$.

19. $\frac{7(x+y+z)^{-1}}{5(x-y-z)^{-1}}$.

CASE V.

158. To reduce dissimilar fractions to similar fractions.

1. Into what parts may $\frac{1}{2}$ of a dollar and $\frac{1}{5}$ of a dollar be divided so that the parts may be of the same size?

2. Into what fractions having the same fractional unit may $\frac{1}{3}$ and $\frac{1}{4}$ be changed?

3. Into what fractions having the same fractional unit may $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{6}$ be changed? Express the resulting fractions in equivalent fractions having their least common denominator.

4. Into what fractions having the same fractional unit may $\frac{1}{2a}$ and $\frac{1}{5a}$ be changed?

5. Into what fractions having the same fractional unit may $\frac{1}{3a}$, $\frac{1}{4a}$, and $\frac{1}{6a}$ be changed? Express the resulting fractions in equivalent fractions having their least common denominator.

6. Express $\frac{1}{2a}$, $\frac{1}{5a}$, and $\frac{1}{10a}$ in equivalent fractions having their least common denominator.

DEFINITIONS.

159. **Similar Fractions** are those which have the same fractional unit.

160. **Dissimilar Fractions** are those which have not the same fractional unit.

Similar fractions have, therefore, a common denominator.

161. When similar fractions are expressed in their lowest terms, they have their **Least Common Denominator.**

162. PRINCIPLES.—1. *A common denominator of two or more fractions is a common multiple of their denominators.*

2. *The least common denominator of two or more fractions is the least common multiple of their denominators.*

EXAMPLES.

1. Reduce $\frac{d}{2ac}$ and $\frac{2c}{3a^2d}$ to similar fractions having their least common denominator.

PROCESS.

$$\frac{d}{2ac}=\frac{d\times 3ad}{2ac\times 3ad}=\frac{3ad^2}{6a^2cd}$$

$$\frac{2c}{3a^2d}=\frac{2c\times 2c}{3a^2d\times 2c}=\frac{4c^2}{6a^2cd}$$

EXPLANATION.—Since the least common denominator of several fractions is the least common multiple of their denominators (Prin. 2), the least common multiple of the denominators $2ac$ and $3a^2d$ must be found, which is $6a^2cd$. The fractions are then reduced to fractions having the denominator $6a^2cd$, according to Case I, by multiplying the numerator and denominator of each fraction by the quotient of $6a^2cd$, divided by the denominator of each of the given fractions. $6a^2cd \div 2ac = 3ad$, the multiplier of the terms of the first fraction. $6a^2cd \div 3a^2d = 2c$, the multiplier of the terms of the second fraction.

RULE.—*Find the least common multiple of the denominators of the fractions for a least common denominator.*

Divide this denominator by the denominator of each fraction, and multiply the terms of the fraction by the quotient.

1. Any multiple of the least common denominator will be a *common* multiple of the denominators.

2. All mixed quantities should be changed to the fractional form, and all fractions to their lowest terms before finding their least common denominator.

Reduce the following to similar fractions having their least common denominator:

2. $\frac{3x}{4}$ and $\frac{5x}{6}$.

3. $\frac{7a}{8}$ and $\frac{5a}{6}$.

4. $\frac{3x^2y}{4}$ and $\frac{2xy^2}{16}$

5. $\frac{2x}{3a}$ and $\frac{4y}{6a}$.

6. $\frac{2b}{3y}$ and $\frac{2c}{6y^2}$.

7. $\frac{3ac}{2x^2y}$ and $\frac{2bd}{3x^2z}$.

8. $\frac{2x-4y}{5x^2}$ and $\frac{3x-8y}{10x}$.

9. $\frac{4a+5b}{3a^2}$ and $\frac{3a+4b}{4a}$.

10. $\frac{3x-2y}{5ac}$ and $\frac{4x-3y}{10a^2c}$.

11. $\frac{3}{2xy}, \frac{4a}{4x^2y}, \frac{5c}{3yx^2z}$.

12. $\frac{c}{4x}, \frac{d}{4xy}, \frac{d}{8x^2y^2}$.

13. $\frac{d}{a^2c}, \frac{c}{3ac^2}, 4$.

14. $\frac{x+y}{4}, \frac{x-y}{2c}, \frac{x^2+y^2}{2a}$.

15. $\frac{x+2}{x-1}, \frac{x-2}{x+1}, \frac{x+3}{x^2-1}$.

16. $\frac{x^2y}{a+b}, \frac{xy}{a-b}, \frac{xy^2}{a^2-b^2}$.

17. $\frac{x+y}{x-y}, \frac{x-y}{x+y}, \frac{x^2+y^2}{x^2-y^2}$.

18. $\frac{x^2-1}{x^2+1}, \frac{x^2+1}{x^2-1}$, and $\frac{x^4+1}{x^4-1}$.

19. $\frac{1}{(a-b)(b-c)}$ and $\frac{1}{(a-b)(a-c)}$.

CLEARING EQUATIONS OF FRACTIONS.

163. 1. Ten is one-half of what number?

2. If one-third of a number is 12, what is the number?

3. If $\frac{1}{2}x$ equals 4, what is the value of x?

4. If $\frac{1}{4}x = 8$, what is the value of x?

5. If both members of an equation are multiplied by the same quantity, how is the equality of the members affected?

6. When $\frac{x}{3} = 6$, what is the resulting equation when each

member is multiplied by 3? How is the equality of the members affected?

7. When $\frac{x}{5} = 10$, what is the resulting equation when each member is multiplied by 5? How is the equality of the members affected?

8. Change into an equation without the fraction $\frac{x}{6} = 5$; $\frac{x}{4} = 6$; $\frac{x}{2} = 12$; $\frac{x}{5} = 20$; $\frac{x}{4} = 10$.

9. How may an equation containing fractions be changed into an equation without fractions?

DEFINITIONS.

164. Clearing an equation of Fractions is changing it into another equation without the fractions.

165. PRINCIPLE.—*An equation may be cleared of fractions by multiplying both members by some multiple of the denominators of the fractions.* (Art. 59, Ax. 3.)

1. Find the value of x in the following $x + \frac{x}{5} = 12$.

PROCESS.

$$x + \frac{x}{5} = 12$$

Clearing of fractions,	$5x + x = 60$
Uniting terms,	$6x = 60$
Therefore,	$x = 10$

EXPLANATION.—Since the equation contains a fraction, it may be cleared of fractions by multiplying both members by the denominator of the fraction (Prin.). The denominator is 5; therefore both members are multiplied by 5, giving as a resulting equation $5x + x = 60$. Uniting similar terms, $6x = 60$; therefore, $x = 10$.

2. Given $x+\frac{x}{3}+\frac{x}{5}+\frac{x}{6}=\frac{153}{10}$, to find the value of x.

PROCESS.

$$x+\frac{x}{3}+\frac{x}{5}+\frac{x}{6}=\frac{153}{10}$$

Clearing of fractions, $30x+10x+6x+5x=459$

Therefore $51x=459$

And, $x=9$

EXPLANATION.—Since the equation may be cleared of fractions by multiplying by some multiple of the denominators (Prin.), this equation may be cleared of fractions by multiplying both members by 3, 5, 6, and 10 successively, or by their product, or by any multiple of 3, 5, 6, and 10.

Since the multiplier will be the smallest when we multiply by the least common multiple of the denominators, for convenience we multiply both members by 30, the least common multiple of 3, 5, 6, and 10. Uniting terms, and dividing, the result is $x=10$.

RULE.—*Multiply both members of the equation by the least common multiple of the denominators.*

1. An equation may also be cleared of fractions by multiplying each member by all the denominators.

2. If a fraction has the minus sign before it, the signs of all the terms of the numerator must be changed when the denominator is removed.

3. Multiplying a fraction by its denominator removes the denominator.

Find the value of x, and verify the result in the following:

3. $x+\frac{x}{5}=24.$

4. $\frac{x}{6}+x=21.$

5. $2x+\frac{x}{3}=28.$

6. $4x+\frac{x}{5}=42.$

7. $3x - \frac{x}{7} = 40.$

8. $x - \frac{x}{6} = 25.$

9. $\frac{4x}{6} - x = -24.$

10. $\frac{3x}{5} + 7x = 38.$

11. $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} = 26.$

12. $\frac{x}{3} + \frac{x}{4} + \frac{x}{6} = 18.$

13. $x + \frac{2x}{3} + \frac{3x}{4} = 29.$

14. $2x + \frac{x}{3} - \frac{x}{4} = 50.$

15. $3x - \frac{2x}{3} - \frac{5x}{6} = 18.$

16. $4x + \frac{x}{3} - \frac{x}{9} = 74.$

17. $3x - \frac{x}{6} + \frac{x}{12} = 70.$

18. $\frac{x}{4} + \frac{x}{6} + \frac{x}{8} = 26.$

19. Given $\frac{x+9}{4} + \frac{2x}{7} = \frac{3x-6}{5} + 3$, to find x.

20. Given $\frac{3x+4}{7} + \frac{4x-51}{47} = 0$, to find x.

21. Given $\frac{3x}{4} + \frac{180-5x}{6} = 29$, to find x.

22. Given $\frac{1}{4}x + \frac{1}{10}x + \frac{1}{8}x = 19$, to find x.

23. Given $\frac{x+3}{2} + \frac{3x}{5} = 2 + \frac{4x-5}{3}$, to find x.

24. Given $\frac{x+2}{5} + \frac{x-1}{7} = \frac{x-2}{2}$, to find x.

25. Given $\frac{7x+2}{10} - 12 = \frac{3x+3}{5} - \frac{x}{2}$, to find x.

26. Given $\frac{3x}{4} + a - \frac{x}{5} = a + \frac{x}{5} + 5\frac{1}{2}$, to find x.

27. Given $\frac{2x+4}{3} - 3\frac{1}{3} = \frac{x-3}{4} + \frac{x+2}{3}$, to find x.

28. Given $\frac{3x-4}{2}=\frac{6x-5}{8}+\frac{3x-1}{16}$, to find x.

29. Given $\frac{2x-5}{6}+\frac{6x+3}{4}=5x-17\frac{1}{2}$, to find x.

30. Given $\frac{5-3x}{4}+\frac{3-5x}{3}=\frac{3}{2}-\frac{5x}{3}$, to find x.

31. Given $\frac{x+3}{2}+\frac{x+4}{3}+\frac{x+5}{4}=16$, to find x.

32. Given $\frac{2x-1}{5}+\frac{6x-4}{7}=\frac{7x+12}{11}$, to find x.

33. Given $\frac{x+1}{3}-\frac{3x-1}{5}=x-2$, to find x.*

34. Given $\frac{x-3}{4}-\frac{x-1}{9}=\frac{x-5}{6}$, to find x.

35. Given $\frac{x}{2}+3=\frac{x}{4}-\frac{x-2}{5}$, to find x.

36. Given $\frac{x-1}{2}+\frac{x-3}{4}-\frac{x-2}{3}=\frac{2}{3}$, to find x.

37. Given $\frac{1-2x}{3}-\frac{4-5x}{4}=-\frac{13}{42}$, to find x.

38. Given $\frac{x+3}{2}-\frac{x-2}{3}-\frac{1}{4}=\frac{3x-5}{12}$, to find x.

39. Given $\frac{4x-2}{11}+4-\frac{3x-5}{13}=5$, to find x.

40. Given $\frac{3x-3}{4}-\frac{3x-3}{3}=\frac{15}{3}-\frac{27+4x}{9}$, to find x.

---

*In clearing this and the following equations of fractions, the signs should be changed, as indicated in Note 2, under the Rule.

41. A spent $\frac{1}{4}$ of his money, and then received \$2. He then spent $\frac{1}{2}$ of what he had, and had \$7 remaining. How much had he at first?

PROCESS.

Let $x =$ the money he had at first.

Then, $\dfrac{x}{4} =$ what he spent at first.

$\dfrac{3x}{4} + 2 =$ what he had after he received \$2.

$\dfrac{1}{2}\left(\dfrac{3x}{4} + 2\right) = \dfrac{3x}{8} + 1 =$ what he spent the second time.

$$\text{Therefore, } \frac{x}{4} + \frac{3x}{8} + 1 + 7 = x + 2$$

Clearing of fractions, $2x + 3x + 8 + 56 = 8x + 16$

Transposing, $2x + 3x - 8x = -48$

$$-3x = -48$$

$$x = 16$$

42. What number is there to which, if $\frac{1}{4}$ of it be added, the sum will be 15?

43. Find a number such that the sum of $\frac{1}{2}$ of it and $\frac{1}{3}$ of it is 15.

44. One-third of A's age plus two-fifths of A's age equals 22 years. How old is he?

45. Three sons were left a legacy, of which the eldest received $\frac{3}{8}$, the second $\frac{1}{5}$, and the third the rest, which was \$200. How much did each receive?

46. A's capital was $\frac{3}{4}$ of B's. If A's had been \$500 less, it would have been but $\frac{1}{2}$ of B's. What was the capital of each?

47. A horse and carriage cost \$420. If the horse cost $\frac{3}{4}$ as much as the carriage, what was the cost of each?

48. A had twice as much money as B, C $1\frac{1}{3}$ times as

much as A, D $\frac{1}{4}$ as much as A, and they all had \$50. How much had each?

49. What number is there, $\frac{1}{5}$ of which is 3 greater than $\frac{1}{6}$ of it?

50. A clerk spent $\frac{1}{3}$ of his salary for board, $\frac{1}{3}$ of the rest for other expenses, and saved annually \$280. What was his salary?

51. There is a number such that, if $\frac{1}{5}$ of it is subtracted from 50, and the remainder multiplied by 4, the result will be 70 less than the number. What is the number?

52. Divide 100 into two parts such that, if $\frac{1}{3}$ of one part be subtracted from $\frac{1}{4}$ of the other, the remainder will be 11.

53. There are two numbers whose difference is 1, such that $\frac{1}{2}$ of the first plus $\frac{1}{5}$ of the first is equal to the sum of $\frac{1}{3}$ of the second and $\frac{1}{4}$ of the second. What are the numbers?

54. Five years ago A's age was $2\frac{1}{3}$ times B's. One year hence it will be $1\frac{4}{5}$ times B's. How old is each now?

55. The difference between two numbers is 20, and $\frac{1}{7}$ of one is equal to $\frac{1}{3}$ of the other. What are the numbers?

56. When the sum of the fourth, fifth, and tenth parts of a certain number is taken from 33 the remainder is nothing. What is the number?

57. The difference between two numbers is 8, and the quotient arising from dividing the greater by the less is 3. What are the numbers?

ADDITION OF FRACTIONS.

166. 1. What is the sum of $\dfrac{5}{9}$ and $\dfrac{3}{9}$? Of $\dfrac{3}{11}$ and $\dfrac{7}{11}$? Of $\dfrac{3a}{6}$ and $\dfrac{2a}{6}$?

2. What is the sum of $\frac{1}{3}$ and $\frac{1}{6}$? $\frac{1}{4}$ and $\frac{1}{8}$? $\frac{2a}{3}$ and $\frac{a}{6}$?

3. What kind of fractions can be added without changing their form?

4. What must be done to dissimilar fractions before they can be added? How are dissimilar fractions made similar?

5. What is the sum of $\frac{2}{3a}$ and $\frac{5}{3a}$? $\frac{6a}{2xy}$ and $\frac{3a}{2xy}$?

6. What is the sum of $\frac{x+y}{3c}$ and $\frac{x-y}{3c}$? Of $\frac{4c}{x+y}$ and $\frac{3c}{x+y}$? Of $\frac{2a}{a+b^2}$ and $\frac{4a}{a+b^2}$?

7. What is the sum of $\frac{3}{2x}$, $\frac{3}{4x}$, and $\frac{5}{6x}$?

167. Principles.—1. *Only similar fractions can be added.*

2. *Dissimilar fractions must be reduced to similar fractions before adding.*

EXAMPLES.

1. What is the sum of $\frac{5a}{6}$, $\frac{3a}{4}$, and $\frac{2b}{9}$?

PROCESS.

$$\frac{5a}{6}+\frac{3a}{4}+\frac{2b}{9}=\frac{30a}{36}+\frac{27a}{36}+\frac{8b}{36}=\frac{57a+8b}{36}; \text{ or, } a+\frac{21a+8b}{36}$$

Explanation.—Since the fractions to be added are dissimilar, they must be made similar before adding.

The least common denominator of the given fractions is 36. $\frac{5a}{6}=\frac{30a}{36}$, $\frac{3a}{4}=\frac{27a}{36}$, and $\frac{2b}{9}=\frac{8b}{36}$. Therefore their sum is $\frac{57a+8b}{36}$, which, expressed as a mixed quantity, is $a+\frac{21a+8b}{36}$.

2. Find the sum of $a+\frac{2ax}{7}$ and $3a+\frac{3x}{z}$.

PROCESS.

$$a+3a=4a$$

$$\frac{2ax}{7}+\frac{3x}{z}=\frac{2axz}{7z}+\frac{3xy}{7z};\text{ or, }\frac{2axz+3xy}{7z}$$

$$\text{Entire sum }=4a+\frac{2axz+3xy}{7z}$$

RULE.—*Reduce the given fractions to similar fractions.*

Add their numerators, and write the sum over the common denominator.

When there are entire or mixed quantities, add the entire and fractional parts separately, and then add their results.

Find the sum of the following:

3. $\frac{x}{y}$ and $\frac{y}{z}$.

4. $\frac{2d}{c}$ and $\frac{3a}{cd}$.

5. $\frac{x}{2z}$ and $\frac{y}{3xz}$.

6. $\frac{x}{3a}$ and $\frac{3y}{6ax}$.

7. $\frac{1}{3ay}$ and $\frac{2x}{3by}$.

8. $\frac{4a}{3xy}$ and $\frac{5a^2}{3ax^2y}$.

9. $\frac{a}{a+x}$ and $\frac{z}{a-z}$.

10. $\frac{1+x}{1-x}$ and $\frac{1-x}{1+x}$.

11. $\frac{1+x^2}{1-x^2}$ and $\frac{1-x^2}{1+x^2}$.

12. $\frac{4x^2}{1-x^4}$ and $\frac{1-x^2}{1+x^2}$.

13. $\frac{x}{x^2-y^2}$ and $\frac{y}{x+y}$.

14. $\frac{2}{1+a}$ and $\frac{a^2+1}{a+a^2}$.

15. Add $\frac{x^2}{x^2-1}$, $\frac{x}{x-1}$, and $\frac{x}{x+1}$.

16. Add $a+\dfrac{1}{a^2-b^2}$ and $2a-b+\dfrac{1}{(a-b)^2}$.

17. Add $\dfrac{a}{a-b}$, $\dfrac{b}{a+b}$, $\dfrac{a-b}{a+b}$, and $\dfrac{ab}{a^2-b^2}$.

18. Add $\dfrac{y^2-2xy-x^2}{x^2-xy}$ and $\dfrac{x}{x-y}$.

19. Add $\dfrac{1}{2(x-1)}$, $\dfrac{1}{2(x+1)}$, and $\dfrac{1}{x^2}$.

20. Add $\dfrac{1+x}{1+x+x^2}$ and $\dfrac{1-x}{1-x+x^2}$.

SUBTRACTION OF FRACTIONS.

168. 1. From $\dfrac{5}{9}$ subtract $\dfrac{3}{9}$. From $\dfrac{7}{11}$ subtract $\dfrac{3}{11}$. From $\dfrac{3a}{6}$ subtract $\dfrac{2a}{6}$. From $\dfrac{3x^2}{8}$ subtract $\dfrac{x^2}{8}$.

2. What is the difference between $\dfrac{1}{3}$ and $\dfrac{1}{6}$? Between $\dfrac{1}{4}$ and $\dfrac{1}{8}$? Between $\dfrac{2a}{3}$ and $\dfrac{a}{6}$? Between $\dfrac{2x}{5}$ and $\dfrac{2x}{10}$?

3. What kind of fractions can be subtracted without changing their form?

4. What must be done to dissimilar fractions before they can be subtracted? How are dissimilar fractions made similar?

5. What is the difference between $\dfrac{5}{3a}$ and $\dfrac{2}{3a}$? Between $\dfrac{6a}{2xy}$ and $\dfrac{3a}{2xy}$? Between $\dfrac{7ax}{3(a+b)}$ and $\dfrac{3ax}{3(a+b)}$?

6. What is the difference between $\frac{x+y}{3c}$ and $\frac{x+2y}{3c}$? Between $\frac{a+4c}{x+y}$ and $\frac{a+3c}{x+y}$? Between $\frac{3ax^2}{(x+y)^2}$ and $\frac{ax^2}{(x+y)^2}$?

7. What is the difference between $\frac{3}{4x}$ and $\frac{3}{2x}$? Between $\frac{3}{4x}$ and $\frac{5}{6x}$? Between $\frac{2}{4a}$ and $\frac{3}{8a}$?

169. PRINCIPLES.—1. *Only similar fractions can be subtracted.*

2. *Dissimilar fractions must be reduced to similar fractions before subtracting.*

EXAMPLES.

1. Subtract $\frac{2a}{7b}$ from $\frac{6b}{11a}$.

PROCESS.

$$\frac{6b}{11a}-\frac{2a}{7b}=\frac{42b^2}{77ab}-\frac{22a^2}{77ab}=\frac{42b^2-22a^2}{77ab}$$

EXPLANATION.—Since the fractions are not similar, before subtracting they must be changed to similar fractions. The least common denominator of the fractions is $77ab$. Therefore, $\frac{6b}{11a}=\frac{42b^2}{77ab}$, and $\frac{2a}{7b}=\frac{22a^2}{77ab}$. Subtracting the numerator of the subtrahend from the numerator of the minuend, the remainder is $\frac{42b^2-22a^2}{77ab}$.

2. From $6a+\frac{3x-2a}{a}$ take $2a+\frac{4a-3x}{x}$.

PROCESS.

$$6a - 2a = 4a$$

$$\frac{3x-2a}{a} - \frac{4a-3x}{x} = \frac{3x^2-2ax}{ax} - \frac{4a^2-3ax}{ax} =$$

$$\frac{3x^2+ax-4a^2}{ax} = 1 + \frac{3x^2-4a^2}{ax}$$

$$\text{Entire remainder} = 4a + 1 + \frac{3x^2-4a^2}{ax}$$

EXPLANATION.—Since the quantities are mixed quantities, the entire quantities and the fractions may be subtracted separately, and the results united.

RULE.—*Reduce the given fractions to similar fractions. Subtract the numerator of the subtrahend from the numerator of the minuend, and place the result over the common denominator.*

When there are entire or mixed quantities, subtract the entire and fractional parts separately, and unite the results.

Subtract:

3. $\frac{3a}{5}$ from $\frac{5a}{6}$.

4. $\frac{2x}{7}$ from $\frac{3x}{5}$.

5. $\frac{4a}{7b}$ from $\frac{3a}{4b}$.

6. $\frac{2x}{4a}$ from $\frac{5x}{9a}$.

7. $\frac{3d}{ax}$ from $\frac{7d}{2ax}$.

8. $\frac{3c}{2ad}$ from $\frac{8c}{ad}$.

Subtract:

9. $\frac{2ab}{3xy}$ from $\frac{5ad}{2xy}$.

10. $\frac{3mn}{4y^2}$ from $\frac{2mn}{4y}$.

11. $\frac{a+b}{3}$ from $\frac{a-b}{2}$.

12. $\frac{3}{a+b}$ from $\frac{2}{a-b}$.

13. $\frac{4}{x-1}$ from $\frac{5}{x+1}$.

14. $\frac{x+1}{x-1}$ from $\frac{x-1}{x+1}$.

15. From $\dfrac{x}{x-3}$ subtract $\dfrac{x}{x+3}$.

16. From $\dfrac{(a+b)^2}{a-b}$ subtract $\dfrac{(a-b)^2}{a+b}$.

17. From $6x+\dfrac{2a-3b}{5a}$ subtract $3x-\dfrac{3a+2b}{6a}$.

18. From $7x+\dfrac{2x}{y}$ subtract $3x-\dfrac{x-3z}{y}$.

Simplify the following expressions:

19. $\dfrac{2x+5y}{x^2y}+\dfrac{4xy-3y^2}{xy^2}-\dfrac{5xy-2y^2}{x^2y^2}$.

20. $\dfrac{3ab-4}{a^2b^2}-\dfrac{6a^2-1}{a^3b}-\dfrac{5b^2+7}{ab^3}$.

21. $\dfrac{x}{1-x}-\dfrac{x^2}{1-x^2}+\dfrac{x}{1+x^2}$.

22. $\dfrac{1}{x-a}+\dfrac{4a}{(x-a)^2}-\dfrac{5a^2}{(x-a)^3}$.

23. $\dfrac{3}{x}-\dfrac{5}{2x-1}-\dfrac{2x-7}{4x^2-1}$.

24. $\dfrac{x^2+y^2}{xy}-\dfrac{x^2}{xy+y^2}-\dfrac{y^2}{x^2+xy}$.

MULTIPLICATION OF FRACTIONS.

CASE I.

170. When the multiplier is an entire quantity.

1. How many fifths are 6 times $\dfrac{3}{5}$? 5 times $\dfrac{2a}{5}$?

2. How many times $\frac{b}{c}$ is 5 times $\frac{3b}{c}$? 7 times $\frac{5b}{c}$? $3a$ times $\frac{3b}{c}$? $3d$ times $\frac{2b}{c}$?

3. How may a fraction be multiplied by an entire quantity?

4. What effect upon a fraction has multiplying its numerator?

5. Express 2 times $\frac{3a}{8}$ in its lowest terms. How may the result be obtained from the terms of $\frac{3a}{8}$?

6. In what other way, besides by multiplying the numerator, may a fraction be multiplied?

7. How much is 3 times $\frac{5a}{6}$? 4 times $\frac{3}{8a}$? 6 times $\frac{5a}{12}$?

8. How much is 5 times $\frac{3a}{7}$? 6 times $\frac{2}{5a}$? 9 times $\frac{3a}{8b}$?

PRINCIPLE.—*Multiplying the numerator, or dividing the denominator of a fraction by any quantity, multiplies the fraction by that quantity.*

EXAMPLES.

1. Multiply $\frac{n}{d}$ by m.

PROCESS.

$$\frac{n}{d} \times m = \frac{mn}{d}$$

EXPLANATION.—Since a fraction is multiplied by multiplying its numerator (Prin.), $\frac{n}{d}$ is multiplied by m by multiplying the n by m. Hence, the product is $\frac{mn}{d}$.

2. Multiply $\frac{2a}{x^2y}$ by x^2.

PROCESS.

$$\frac{2a}{x^2y} \times x^2 = \frac{2a}{x^2y \div x^2} = \frac{2a}{y}$$

EXPLANATION.—Since a fraction may also be multiplied by dividing its denominator (Prin.), $\frac{2a}{x^2y}$ may be multiplied by x^2 by dividing the denominator by x^2, since it is a factor of the denominator. Hence, the result is $\frac{2a}{y}$.

RULE.—*Multiply the numerator, or divide the denominator by the multiplier.*

It is often best to indicate the multiplication, and then cancel equal factors from both numerator and denominator.

Multiply:

3. $\frac{x}{z}$ by z.

4. $\frac{x^2}{yz}$ by x.

5. $\frac{xy}{a}$ by a.

6. $\frac{a^2b}{cd}$ by cb.

7. $\frac{m^2n}{ab}$ by n^2a.

8. $\frac{r^2s}{a^2b}$ by a^2bc.

9. $\frac{axy}{c^2d}$ by c^2d^2.

Multiply:

10. $\frac{3ax}{x+y}$ by $2ax$.

11. $\frac{4by}{3(a+x)}$ by $2ay$.

12. $\frac{5a^2x^2}{2(c+d)^2}$ by $(c+d)$.

13. $\frac{3a+b}{3(x+y)}$ by $(x+y)$.

14. $\frac{2az}{4(m-n^2)}$ by $(m-n^2)$.

15. $\frac{4c^2d^2}{3(a+b)}$ by $9(a+b)^2$.

16. $\frac{m+n}{m^2+n^2}$ by $(m+n)$.

17. Multiply $\frac{3rs^2}{2(y+z)}$ by $3(y+z)$.

18. Multiply $\frac{3(x^2+2y)}{x^2-y^2}$ by $(x+y)$.

CASE II.

171. When the multiplier is a fraction.

1. How much is one-half of $\frac{4}{5}$ of a dollar? One-half of $\frac{4}{5}$ of a? One-half of $\frac{6}{7}$ of b? One-half of $\frac{6b}{7}$?

2. How much is $\frac{1}{3}$ of $\frac{6}{7}$? $\frac{1}{4}$ of $\frac{8a}{9}$? $\frac{1}{5}$ of $\frac{10x}{11}$?

3. What may be done to the numerator of a fraction to obtain $\frac{1}{2}$ of the fraction? To obtain $\frac{1}{3}$ of it? $\frac{1}{4}$ of it? To find any part of it?

4. If $\frac{1}{2}$ is divided into two equal parts, what will be the value of each part? How much is $\frac{1}{2}$ of $\frac{1}{2}$? $\frac{1}{2}$ of $\frac{1}{4}$? $\frac{1}{2}$ of $\frac{1}{3}$? $\frac{1}{3}$ of $\frac{1}{4}$? $\frac{1}{3}$ of $\frac{2}{5}$? $\frac{1}{5}$ of $\frac{a}{4}$? $\frac{1}{3}$ of $\frac{2x}{7}$? $\frac{1}{4}$ of $\frac{3x}{7y}$?

5. In what other way, besides by dividing the numerator, may $\frac{1}{2}$ of a fraction be found, or the fraction be divided by 2? How may $\frac{1}{3}$ of it be found? $\frac{1}{4}$ of it? $\frac{1}{5}$ of it? Any part of it?

PRINCIPLE.—*Dividing the numerator, or multiplying the denominator of a fraction by any quantity, divides the fraction by that quantity.*

EXAMPLES.

1. What is the product of $\frac{a}{b}$ multiplied by $\frac{c}{d}$?

PROCESS.

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

EXPLANATION.—To multiply $\frac{a}{b}$ by $\frac{c}{d}$ is to find c times $\frac{1}{d}$ part of $\frac{a}{b}$. $\frac{1}{d}$ part of $\frac{a}{b} = \frac{a}{bd}$ (Prin.), and c times $\frac{a}{bd} = \frac{ac}{bd}$.

RULE.—*Multiply the numerators together for the numerator of the product, and the denominators together for its denominator.*

1. Reduce all entire and mixed quantities to the fractional form before multiplying.

2. Entire quantities may be expressed in the form of a fraction by writing 1 as a denominator. Thus, a may be written $\frac{a}{1}$.

3. When possible, *cancel* equal factors from numerator and denominator.

Multiply:

2. $\frac{a}{b}$ by $\frac{n}{m}$.

3. $\frac{3ac}{4b}$ by $\frac{4x}{2ay}$.

4. $\frac{5x^2y^2}{a^2x^2}$ by $\frac{3ax^2}{2ay^2}$.

5. $\frac{a^4b^4}{2a^2y^n}$ by $\frac{a^2x}{xy^n}$.

Multiply:

6. $\frac{x-y}{a^2}$ by $\frac{a^3y}{2x}$.

7. $\frac{x+y}{10}$ by $\frac{ax}{3(x+y)}$.

8. $\frac{2a+3b}{2x}$ by $\frac{2x}{4b}$.

9. $\frac{x^2-a^2}{xy}$ by $\frac{xy}{x+a}$.

Multiply:

10. $\dfrac{a}{x-y}$ by $\dfrac{b}{x+y}$.

11. $\dfrac{x^2-xy}{a+c}$ by $\dfrac{a+c}{x-y}$.

12. $\dfrac{x+y}{x-y}$ by $\dfrac{x^2-y^2}{(x+y)^2}$.

13. $\dfrac{x^4-y^4}{a^2x^2}$ by $\dfrac{ax^3}{x^2+y^2}$.

Multiply:

14. $\dfrac{c}{x^2-y^2}$ by $\dfrac{d}{x^2+y^2}$.

15. $\dfrac{3a^2}{5x-15}$ by $\dfrac{15x-45}{2a}$.

16. $\dfrac{4ax}{2+3x}$ by $\dfrac{12+18x}{8x^2}$.

17. $\dfrac{3a^2x^2}{4xy}$ by $\dfrac{3xy}{4a^2(x+y)}$.

Simplify the following:

18. $\dfrac{(a-x)^2}{2c}\times\dfrac{3ax}{a-x}\times\dfrac{2}{a(a-x)}$.

19. $\dfrac{x+y}{(x-y)^2}\times\dfrac{x-y}{(x+y)^2}\times\dfrac{x-y}{x+y}$.

20. $\dfrac{x^2+4x}{x^2-3x}\times\dfrac{6x^2-18x}{4x^2+16x}$.

21. $\dfrac{x^2-11x+30}{x^2-6x+9}\times\dfrac{x^2-3x}{x^2-5x}$.

22. $\dfrac{x^2+x-2}{x^2-7x}\times\dfrac{x^2-13x+42}{x^2+2x}$.

23. $\dfrac{x^2+3x+2}{x^2-5x+6}\times\dfrac{x^2-7x+12}{x^2+x}$.

24. $\left(x+\dfrac{xy}{x-y}\right)\times\left(y-\dfrac{xy}{x+y}\right)$.

25. $\left(4+\dfrac{2x}{3c}\right)\times\left(2-\dfrac{2x}{6c}\right)$.

26. $\left(\dfrac{a}{x+a}-\dfrac{x}{x-a}\right)\times\left(\dfrac{x}{a}-\dfrac{a}{x}\right)$.

27. $\dfrac{a^4-b^4}{a^2-2ab+b^2}\times\dfrac{a-b}{a^2+ab}$.

28. $\dfrac{x(a-x)}{a^2+2ax+x^2}\times\dfrac{a(a+x)}{a^2-2ax+x^2}$.

29. $\dfrac{a^2+ab}{a^2+b^2}\times\left(\dfrac{a}{a-b}-\dfrac{b}{a+b}\right)$.

30. $\dfrac{a}{a-b}\times\dfrac{a}{a+b}\times\dfrac{a^2-b^2}{a^2}$.

31. $\dfrac{a-a^2x^2}{b+by}\times\dfrac{b-by^2}{ax+ax^2}$.

DIVISION OF FRACTIONS.

CASE I.

172. When the divisor is an entire quantity.

1. If $\frac{3}{4}$ is divided into 3 equal parts, what is the value of each part? What is the value of $\frac{3}{4}\div 3$? Of $\frac{3a}{5}\div 3$? Of $\frac{5x}{7}\div 5$? Of $\frac{8x}{9}\div 4$?

2. What is the value of $\frac{15a}{23}\div 5a$? Of $\frac{12xy}{2c}\div 6xy$? $\frac{20ab}{13xy}\div 10ab$? Of $\frac{21a^2xy}{17c}\div 7a^2xy$?

3. How may a fraction be divided by an entire quantity when the divisor is a factor of the numerator?

4. If $\frac{1}{2}$ is divided into 2 equal parts, what is the value of each part? What is the value of $\frac{1}{2} \div 2$? Of $\frac{1}{4} \div 2$? Of $\frac{1}{5} \div 2$? Of $\frac{a}{3} \div 2$? Of $\frac{a^2x}{4} \div 2$?

5. What is the value of $\frac{3}{5} \div 4$? Of $\frac{3x}{4y} \div 5$? Of $\frac{5x}{6ay} \div 4$? Of $\frac{3xy}{5ac} \div 5a$? Of $\frac{4xy}{7ab} \div 2ab$?

6. How may a fraction be divided by an entire quantity when the divisor is *not* a factor of the numerator?

7. What is the value of $\frac{3x}{4ac} \div 3x$? Of $\frac{4ac}{5xy} \div 2a$? Of $\frac{x}{4ab} \div 2a$? Of $\frac{ay}{3bc} \div 2x$?

EXAMPLES.

1. Divide $\frac{ab}{c}$ by a.

PROCESS.

$$\frac{ab}{c} \div a = \frac{ab \div a}{c} = \frac{b}{c}$$

OR,

$$\frac{ab}{c} \div a = \frac{ab}{c \times a} = \frac{b}{c}$$

EXPLANATION.—Since dividing the numerator of a fraction divides the fraction (Art.), the fraction $\frac{ab}{c}$ may be divided by a by dividing the numerator by a. Or,

Since multiplying the denominator divides the fraction (Art.), the fraction may be divided by multiplying the denominator by a. The result by both processes is the same, $\frac{b}{c}$.

RULE.—*Divide the numerator or multiply the denominator by the entire quantity.*

It is often best to indicate the division and cancel common factors.

Divide:

2. $\frac{24a^2b}{11}$ by $3a^2$.

3. $\frac{35xy^2}{4ab}$ by $7xy$.

4. $\frac{4x^2y^3}{27az}$ by $6x^2y$.

5. $\frac{25xyz}{17abc}$ by $5xy$.

Divide:

6. $\frac{20a^2b^2}{17xy}$ by $5a^2x$.

7. $\frac{24x^2y^3}{15a^2z^2}$ by $12xyz$.

8. $\frac{25a^2bc}{15c^2dx}$ by $5abc$.

9. $\frac{x^2y+xyz}{v}$ by xy.

10. Divide $\frac{33x^2y^2z}{15a^2b^2}$ by $11x^2y^2z^2$.

11. Divide $\frac{a^2-c^2}{1+x}$ by $a+c$.

12. Divide $\frac{a^2+4x+4}{d+c}$ by $a+z$.

13. Divide $\frac{5(x+y)^2}{x-y}$ by $x+y$.

14. Divide $\frac{ab+cd}{a+c}$ by $a-c$.

15. Divide $\frac{3an+cm}{x^2-y^2}$ by x^2+y^2.

16. Divide $\frac{4ax+4by}{c^2+d^2}$ by $a+b$.

17. Divide $\frac{5xy+5xz}{a+n}$ by $5(x+z)$.

CASE II.

173. When the divisor is a fraction.

1. How many times is $\frac{1}{8}$ contained in 1? $\frac{2}{8}$? $\frac{3}{8}$? $\frac{4}{8}$?

2. How many times is $\frac{a}{8}$ contained in a? $\frac{2a}{8}$? $\frac{3a}{8}$? $\frac{4a}{8}$?

3. What is the value of $1 \div \frac{1}{8}$? $1 \div \frac{1}{9}$? $1 \div \frac{2}{8}$? $1 \div \frac{2}{9}$?

4. How many times is $\frac{a}{4}$ contained in $\frac{3a}{4}$? $\frac{a}{4}$ in $\frac{7a}{4}$? $\frac{a}{3}$ in $\frac{2a}{3}$? $\frac{a}{5}$ in $\frac{2a}{5}$? $\frac{2a}{7}$ in $\frac{4a}{7}$? $\frac{3a}{8}$ in $\frac{6a}{8}$?

5. How many times is $\frac{a}{4}$ contained in $\frac{a}{2}$? $\frac{a}{8}$ in $\frac{a}{2}$? $\frac{a}{16}$ in $\frac{a}{2}$? $\frac{a}{6}$ in $\frac{a}{3}$? $\frac{a}{9}$ in $\frac{a}{3}$?

EXAMPLES.

1. What is the value of $\frac{a}{c} \div \frac{b}{d}$?

PROCESS.

$$\frac{a}{c} \div \frac{b}{d} = \frac{a}{c} \times \frac{d}{b} = \frac{ad}{bc}$$

OR,

$$\frac{ad}{cd} \div \frac{bc}{cd} = \frac{ad}{bc}$$

EXPLANATION.—$\frac{1}{d}$ is contained in 1, d times; and $\frac{b}{d}$ is contained in 1, $\frac{1}{b}$ part of d times, or $\frac{d}{b}$ times.

And, since $\frac{b}{d}$ is contained in 1, $\frac{d}{b}$ times, it will be contained in $\frac{a}{c}$, $\frac{a}{c}$ times $\frac{d}{b}$, or $\frac{ad}{bc}$ times. Or, $\frac{a}{c}$ is equal to $\frac{ad}{cd}$, and $\frac{b}{d}$ is equal to $\frac{bc}{cd}$; $\frac{bc}{cd}$ is contained in $\frac{ad}{cd}$, $\frac{ad}{bc}$ times.

Rule.—*Multiply the dividend by the divisor inverted.*

1. Change entire and mixed quantities to the fractional form.
2. When possible, use cancellation.

Divide:

2. $\frac{a}{b}$ by $\frac{x}{y}$.

3. $\frac{ax}{by}$ by $\frac{cd}{3y}$.

4. $\frac{3x^2y}{4a^2c}$ by $\frac{2xy}{3ac}$.

5. $\frac{4a^3x}{6dy^2}$ by $\frac{2a^2x^2}{8a^2y}$.

6. $\frac{7x^2y}{3ad}$ by $\frac{2xy^2}{3a^2d}$.

7. $\frac{5x^2y^3z}{6a^2b^2c}$ by $\frac{10xy^3z^2}{8ab^2c^2}$.

Divide:

8. $\frac{4ayz}{8bcd}$ by $\frac{6a^2y^2z^2}{16bcd}$.

9. $\frac{axyz}{cmn}$ by $\frac{8ax^2y}{dmn}$.

10. $\frac{mny^2}{abc}$ by $\frac{m^3n^2y^2}{a^2b^2c^2}$.

11. $\frac{5xy}{a-x}$ by $\frac{10xy}{a^2-x^2}$.

12. $\frac{2ax+x^2}{a^3-x^3}$ by $\frac{x}{a-x}$.

13. $\frac{m^2-n^2}{6}$ by $\frac{3m+n}{12}$.

14. Divide $\frac{d^2+2cd+c^2}{12}$ by $\frac{d+c}{8}$.

15. Divide $\frac{8a^3}{a^3-b^3}$ by $\frac{4a^2}{a^2+ab+b^2}$.

16. Divide $\frac{4}{x^3+y^3}$ by $\frac{8}{x+y}$.

17. Divide $\frac{a^4-b^4}{2ab}$ by $\frac{a^2-b^2}{4a^2b^2}$.

18. Divide $a+\frac{c}{d}$ by $\frac{x}{y}$.

19. Divide $\frac{c+dy}{2y}$ by xy.

20. Divide $\dfrac{1}{x^2-17x+30}$ by $\dfrac{1}{x-15}$.

21. Divide $\dfrac{a}{x^2-5x-6}$ by $\dfrac{b}{x^2+x}$.

22. Divide $\dfrac{2x}{x^2-7x}$ by $\dfrac{3x^2}{x^2-13x+42}$.

23. Divide $\dfrac{3ax}{x^2-5x}$ by $\dfrac{4ay}{x^2-11x+30}$.

24. Divide $6x^2-\dfrac{1}{4}$ by $2x+\dfrac{1}{4}$.

25. Divide $a^2+\dfrac{a}{2}$ by $a+\dfrac{1}{2}$.

26. Divide $x^2+\dfrac{y}{x}$ by $y+\dfrac{x}{y}$.

27. Divide $\dfrac{1}{a^2-x^2}$ by $\dfrac{(a+x)^2}{(a-x)^2}$.

28. Divide $\dfrac{b^3+1}{ab^3}$ by $(b+\dfrac{1}{b}-1)$.

29. Divide $\dfrac{1}{a^2-b^2}$ by $\dfrac{5}{a+b}$.

30. Divide $\dfrac{a^2-x^2}{by}$ by $\dfrac{a^3y-ax^2y}{b(y^2-y)}$.

31. Divide $\dfrac{x^4-y^2}{a+xy}$ by $\dfrac{abc(x^2-y)}{a^2x^2-x^4y^2}$.

174. Expressions which have a fraction in either the numerator or denominator, or in both, are called **Complex Fractional Forms.** They are simply expressions of division.

32. Find the value of the expression $\dfrac{\frac{a}{b}}{\frac{c}{d}}$.

PROCESS.

$$\frac{\frac{a}{b}}{\frac{c}{d}}=\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}$$

EXPLANATION.—Fractional forms are simply expressions of division; and, therefore, the given fractional form is the same as though it were written $\frac{a}{b}\div\frac{c}{d}$. Performing the division according to the principles already given the quotient is, $\frac{ad}{bc}$.

Find the value of the following:

33. $\dfrac{x+\frac{a}{c}}{x+\frac{b}{d}}$.

34. $\dfrac{a^2+\frac{x}{3}}{4+\frac{x}{5}}$.

35. $\dfrac{3a^2-3y^2}{\frac{a+y}{3}}$.

36. $\dfrac{\frac{4x-4y}{5ab}}{\frac{5x-3y}{5xy}}$.

37. $\dfrac{\frac{x+y}{4ax}}{\frac{x^2-y^2}{8ax^2}}$.

38. $\dfrac{4a^2-4x^2}{\frac{a+x}{a-x}}$.

39. $\dfrac{x+\frac{2d}{3ac}}{x+\frac{3d}{2ac}}$.

40. $\dfrac{x^2-\frac{y^2}{2}}{\frac{x-3y}{2}}$.

41. $\dfrac{xy-\frac{3x}{ac}}{\frac{ac}{x}+2c}$.

REVIEW OF FRACTIONS.

175. Reduce to their lowest terms:

1. $\dfrac{x^3-6x^2+11x-6}{x^3-2x^2-x+2}$.

2. $\dfrac{m^3+m^2+m-3}{m^3+3m^2+5m+3}$.

3. $\dfrac{x^4-x^3-4x^2-x+1}{4x^3-3x^2-8x-1}$.

4. $\dfrac{a^3-7a^2+16a-12}{3a^3-14a^2+16a}$.

Find the value of the following:

5. $\dfrac{x}{x+1}-\dfrac{x}{1-x}+\dfrac{x^2}{x^2-1}$.

6. $\dfrac{3+2x}{2-x}-\dfrac{2-3x}{2+x}+\dfrac{16x-x^2}{x^2-4}$.

7. $\dfrac{1}{(x-y)(y-z)}+\dfrac{1}{(y-x)(x-z)}+\dfrac{1}{(x-z)(x-y)}$.

8. $\dfrac{1}{a(a-b)(a-c)}+\dfrac{1}{b(b-a)(b-c)}+\dfrac{1}{c(c-a)(c-b)}$.

9. $\left(1+\dfrac{x+y}{x-y}\right)\times\left(1-\dfrac{x-y}{x+y}\right)$.

10. $\left(x-y+\dfrac{x^2+y^2}{x+y}\right)\times\left(x-y+\dfrac{x^2+y^2}{x+y}\right)$.

11. $\left(a^2+1+a\right)\times\left(1-\dfrac{1}{a}+\dfrac{1}{a^2}\right)$.

12. $\left(\dfrac{a+1}{a-1}-\dfrac{a-1}{a+1}\right)\div\dfrac{x}{a-1}$.

13. $\left(\dfrac{2x}{x-1}\right)\times\left(1+\dfrac{x-1}{x+1}\right)\div\left\{\left(1-\dfrac{x-1}{x+1}\right)\times\left(1+\dfrac{x+1}{x-1}\right)\right\}$

14. $\dfrac{1+\dfrac{a-x}{a+x}}{1-\dfrac{a-x}{a+x}}\div\dfrac{1+\dfrac{a^2-x^2}{a^2+x^2}}{1-\dfrac{a^2-x^2}{a^2+x^2}}$.

SIMPLE EQUATIONS.

176. Review.—1. Definition of an **Equation.**

2. Definition of **Members of an Equation.**

3. Definition of **First Member; Second Member.**

4. Definition of **Clearing of Fractions.**

5. Definition of **Transposing.**

6. Definition of an **Axiom.**

7. Definition of a **Statement of a Problem.**

8. Definition of a **Solution of a Problem.**

DEFINITIONS.

177. The **Degree** of an equation is determined from the highest number of factors of unknown quantities contained in any term.

Thus, $x+b=c$, $3ax+y=n$, $4b^2x+3a^3x=a$, are equations of the *first* degree.

$x^2+a=c$, $bx^2+3y=d$, $x+xy=7$, $axy+3y^2=n$, are equations of the *second* degree.

$x^3=a$, $x^2y=a$, $xy^2=a$, $x+x^2+x^3=a$, are equations of the *third* degree.

178. An equation of the *first* degree is called a **Simple Equation.**

179. An equation of the *second* degree is called a **Quadratic Equation.**

180. An equation of the *third* degree is called a **Cubic Equation.**

181. An equation in which all the known quantities are expressed by figures, is called a **Numerical Equation.**

182. An equation in which some or all of the known quantities are expressed by letters, is called a **Literal Equation.**

EXAMPLES.

Find the value of x in the following:

1. $4x - \dfrac{x+2}{2} = 3x + 3.$

2. $x - \dfrac{3x+4}{3} = \dfrac{x}{9} + \dfrac{x-12}{6}.$

3. $\dfrac{6x-8}{2} + 2 = x - \dfrac{5-2x}{4}.$

4. $x - 3 - \dfrac{x+2}{8} = \dfrac{x}{3}.$

5. $\dfrac{15x}{4} = 2\tfrac{1}{4} - \dfrac{3-x}{2}.$

6. $\dfrac{x}{3} - x = \dfrac{x-1}{11} - 9.$

7. $\dfrac{9x}{7} - \dfrac{x+3}{5} = 2x - 21.$

8. $\dfrac{ax-b}{c} + a = \dfrac{x+ac}{c}.$

9. $ax - \dfrac{3a-bx}{2} = \dfrac{1}{4}.$

10. $\dfrac{x}{a} - b = \dfrac{c}{d} - x.$

11. $\dfrac{3x-5}{2}-12=\dfrac{4-2x}{3}-x.$

12. $\dfrac{x}{a-1}-\dfrac{x}{a+1}=b.$

13. $\dfrac{x^2+2ax+a^2}{x+a}=\dfrac{4ab}{16b}.$

14. $2-2x=\dfrac{x+8}{4}-\dfrac{x+6}{3}.$

15. $\dfrac{4x}{5}+\dfrac{3b}{2}=\dfrac{a}{6}+\dfrac{12b}{2}.$

16. $\dfrac{x}{a}-a=\dfrac{a}{c}-\dfrac{x}{c-a}.$

17. $\dfrac{2x-8}{4}+\dfrac{x}{3}=30-\dfrac{x+32}{2}.$

18. $10-\dfrac{3x+4}{3}=2x-3\tfrac{1}{3}.$

19. $4+10x+5-6x\left(\dfrac{1}{x}-\dfrac{1}{3}\right)=27.$

20. $x-\dfrac{x-2}{3}+\dfrac{x-4}{5}=7+\dfrac{x-5}{6}.$

21. $\dfrac{x}{3}-\dfrac{x^2-5x}{3x-7}=\dfrac{2}{3}.$

22. $\dfrac{3}{x+1}-\dfrac{x+1}{x-1}=\dfrac{x^2}{1-x^2}.$

23. $\dfrac{2x^4+2x^3-9x^2+12}{x^2+3x-4}=2x^2-4x-3.$

24. $\dfrac{2}{3}+\dfrac{3x-3}{4}-\dfrac{3x-4}{3}=6-\dfrac{27+4x}{9}.$

25. $\frac{9x+20}{36}-\frac{x}{4}=\frac{4x-12}{5x-4}$.

26. $\frac{2x+8}{5}+\frac{x}{2}-8=\frac{x-\frac{4x-9}{3}}{6}-8\frac{5}{6}$.

27. $\frac{a+b}{x-c}=\frac{a}{x-a}+\frac{b}{x-b}$.

28. $\frac{6x+13}{15}-\frac{9x+15}{5x-25}+3=\frac{2x+15}{5}$.

29. $\frac{x-a}{a-b}-\frac{x+a}{a+b}=\frac{2ax}{a^2-b^2}$.

30. $x+3+\frac{3(x+3)}{7}=\frac{3(x+3)}{2}-\frac{1}{2}$.

PROCESS.

$$x+3+\frac{3(x+3)}{7}=\frac{3(x+3)}{2}-\frac{1}{2}$$
$$y+\frac{3y}{7}=\frac{3y}{2}-\frac{1}{2}$$
$$14y+6y=21y-7$$
$$y=7$$
$$x+3=7$$
$$x=4$$

EXPLANATION.—When the same expression is found in several terms, the process may be shortened by *substitution*. Thus, y is substituted for $x+3$. The value of y is found to be 7. Therefore, $x+3=7$, and $x=4$.

31. $x+6-\frac{3(x+6)}{4}=\frac{1}{3}(x+6)-6$.

32. $x-7+\frac{x-7}{2}+\frac{3(x-7)}{4}=2\frac{1}{4}$.

33. $\frac{3(x+4)}{2}+\frac{x+4}{4}-\frac{3(x+4)}{5}=11\frac{1}{2}$.

34. $\dfrac{2(x-3)}{3}+2(x-3)=5-\dfrac{x-3}{4}.$

35. $\dfrac{21-3x}{3}-\dfrac{2(2x+3)}{9}=6-\dfrac{5x+1}{4}.$

36. $x-4-\dfrac{3x-5}{2}=8-\dfrac{2x-4}{3}.$

37. $\dfrac{x-12}{x-7}+\dfrac{x-4}{x-12}=2+\dfrac{7}{x-7}.$

38. $\dfrac{2x-10}{4}-\dfrac{x-8}{5}-\dfrac{x-5}{2}=x-14.$

39. $\dfrac{2x-5}{3}-\dfrac{x-3}{4}=x-2-\dfrac{x-1}{2}.$

40. $\dfrac{x}{5}-\dfrac{2x-14}{3}=4\frac{1}{3}-\dfrac{x}{2}.$

41. $\dfrac{3x}{4}-\dfrac{3x-11}{2}=6x-\dfrac{20x+13}{4}.$

42. $\dfrac{7x+8}{3x-1}-8=\dfrac{27x-36}{3x-1}+4.$

43. $\dfrac{3x-3}{4}-\dfrac{3x-4}{3}=\dfrac{16}{3}-\dfrac{27+4x}{9}.$

44. $\dfrac{6x+18}{13}-\dfrac{11-3x}{36}=5x-43\frac{1}{6}-\dfrac{13-x}{12}-\dfrac{21-2x}{18}.$

45. $\dfrac{4x+3}{9}=\dfrac{8x+19}{18}-\dfrac{7x-29}{5x-12}.$

46. $\dfrac{ab+x}{b^2}-\dfrac{b^2-x}{a^2b}=\dfrac{x-b}{a^2}-\dfrac{ab-x}{b^2}.$

47. $(a+x)(b+x)-a(b+c)=\dfrac{a^2c}{b}+x^2.$

48. $\frac{3ax-2b}{3b}-\frac{ax-a}{2b}=\frac{ax}{b}-\frac{2}{3}.$

49. $\frac{x}{a-b}-\frac{2+x}{a+b}=\frac{c}{a^2-b^2}+\frac{d}{a-b}.$

50. $\frac{x-ab}{3}-\frac{5b^2}{6}+\frac{x-b^2}{4}=\frac{7a^2}{12}+\frac{11ab}{6}-\frac{x-a^2}{2}.$

PROBLEMS.

183. Directions for Solving.—*Represent one of the unknown quantities by x, and from the conditions of the problem find an expression for each of the other quantities given.*

Find from the problem two expressions that are equal, and express them as an equation.

Solve the equation.

51. When the half of a certain number is added to the number, the sum is as much more than 60 as the number is less than 65. What is the number?

52. The difference between two numbers is 8, and the quotient arising from dividing the greater by the less is 3. What are the numbers?

53. A man left one-half of his property to his wife, one-sixth to his children, a twelfth to his brother, and the rest, which was $600, to charitable purposes. How much property had he?

54. Find two numbers whose sum is 70, such that the first, divided by the second, gives a quotient of 2 and a remainder of 1.

55. Out of a cask of wine, one-fifth part had leaked away. Afterward, 10 gallons were drawn out, when the cask was found to be two-thirds full. How much did it hold?

56. A can do a piece of work in 5 days, and B can do the same work in 6 days. How long will it take both working together to do it?

SOLUTION.

Let x represent the number of days it will take both to do it.

$\frac{1}{x}=$ the part both can do in a day.

$\frac{1}{5}=$ the part of the work which A can do in a day.

$\frac{1}{6}=$ the part of the work which B can do in a day.

$$\text{Therefore, } \frac{1}{5}+\frac{1}{6}=\frac{1}{x}$$
$$6x+5x=30$$
$$11x=30$$
$$x=2\tfrac{8}{11}$$

57. A can do a piece of work in 9 days, and B can do the same in 10 days. How long will it take both to do it?

58. A can do a piece of work in 5 days, B in 7 days, and C in 9 days. In how many days can they all together do it?

59. Two pipes empty into a cistern. One can fill it in 8 hours, and the other in 9 hours. How soon will it be filled, if both empty into it at the same time?

60. A cistern can be filled by a pipe in 3 hours, and emptied by another pipe in 4 hours. How much time will be required to fill the cistern if both are running?

61. A fish was caught whose tail weighed 9 pounds. His head weighed as much as his tail and half his body, and his body weighed as much as his head and tail. How much did the fish weigh?

62. Of a detachment of soldiers, $\frac{2}{3}$ are on duty, $\frac{1}{8}$ of them sick, $\frac{1}{5}$ of the remainder absent on leave, and the rest, 380, have deserted. How many were there in the detachment?

63. A person spends one-fourth of his annual income for his board, one-third for clothes, one-twelfth for other expenses, and saves \$500. What is his income?

Fractions may be avoided in this and similar examples, by letting some number of times x, which is a multiple of the denominators, represent the number sought. Thus, in the above example, let $12x$ represent the annual income.

SOLUTION.

Let $12x =$ his annual income.
$3x =$ what he paid for board.
$4x =$ what he paid for clothes.
$x =$ what he paid for his other expenses.

$$3x + 4x + x + 500 = 12x$$
$$4x = 500$$
$$x = 125$$
$$12x = 1500, \text{ his income}$$

64. A farm of 392 acres was divided among four heirs, so that A had four-fifths as much as B, C as much as A and B, and D one-half as much as A and C. What was the share of each?

65. A farmer wishes to mix 300 bushels of provender, containing rye, corn, and oats, so that the mixture may contain $\frac{2}{3}$ as much oats as corn, and $\frac{1}{2}$ as much rye as oats. How many bushels of each should he use?

66. Into what two parts can the number 204 be divided, such that $\frac{2}{5}$ of the greater being taken from the less, the remainder will be equal to $\frac{3}{7}$ of the less subtracted from the greater?

67. A man spent \$14 more than $\frac{3}{7}$ of his money, and had \$6 more than $\frac{1}{8}$ of it left. How much had he at first?

68. A merchant lost $\frac{1}{5}$ of his capital during the first year. The second year he gained $\frac{3}{8}$ as much as he had left at the end of the first. The third year he gained $\frac{3}{11}$ of what he had at the close of the second, making his capital \$7000. What was his original capital?

69. An officer wished to arrange his men in a solid square. He found by his first arrangement that he had 39 men over. He then increased the number on a side by 1 man, and found he needed 50 men to complete the square. How many men had he?

SOLUTION.

Let $x =$ the number of men in each side in the first arrangement.

Then $x^2 =$ the number of men in the first square.

$x+1 =$ the number of men in each side in the second arrangement.

$(x+1)^2 =$ the number of men in the second square.

$x^2+39 =$ the entire number of men.

$(x+1)^2-50 =$ the entire number of men.

$$\text{Therefore, } (x+1)^2-50=x^2+39$$
$$x^2+2x+1-50=x^2+39$$
$$2x=88$$
$$x=44$$
$$x^2+39=1975$$

70. A regiment of troops was drawn up in a solid square with a certain number on a side, when it was found that there were 295 men left. Upon arranging in a larger square in which each rank contained 5 men more, there were none left. How many men were there in the regiment?

71. A colonel, upon attempting to draw up his troops in the form of a solid square, found that he had 31 men over. If he had increased the side of the square by 1 man there would have been a deficiency of 24 men. How many men were there in the regiment?

72. A person in purchasing sugar found that if he bought sugar at 11 cents he would lack 30 cents of having money enough to pay for it; so he bought sugar at $10\frac{1}{2}$ cents, and had 15 cents left. How many pounds did he buy?

73. Into what two parts may the number 56 be divided, so that one may be to the other as 3 to 4?

SOLUTION.

Since one number is to the other as 3 to 4, one is $\frac{3}{4}$ of the other. Therefore, to avoid fractions

$$\begin{aligned} \text{Let } 4x &= \text{one part.} \\ \text{Then } 3x &= \text{the other part.} \\ 4x + 3x &= 56 \\ 7x &= 56 \\ x &= 8 \\ 4x &= 32, \text{ one part} \\ 3x &= 24, \text{ the other part} \end{aligned}$$

74. Find two numbers which are to each other as 5 to 7, and whose sum is 72.

75. A's age is to B's as 3 to 8, and the sum of their ages is 44 years. How old is each?

76. An estate of \$15000 was divided between two sons, so that the elder's share was to the younger's as 8 to 7. What was the share of each?

77. A sum of money was divided between A and B, so that the share of A was to that of B as 5 to 3. The share of A also exceeded $\frac{5}{9}$ of the whole sum by \$50. What was the share of each?

78. A and B began to play together with equal sums of money. A won \$20, but afterward lost half of all he then had, when he found that he had just half as much as B. How much had each at first?

79. A lady distributed \$252 among some poor people, giving to the men \$12 each, the women \$6 each, and the children \$3 each. The number of women was 2 less than twice the number of men, and the number of children was 4 less than 3 times the number of women. To how many persons did she give the money?

80. A person bought a number of apples at the rate of 5 for 2 cents. He sold half of them at 2 for a cent, and the remainder at 3 for a cent, gaining 1 cent. How many did he buy?

81. A merchant engaged in business with a certain capital. His gain the first year lacked \$1000 of being as much as his original capital. His gain the second year lacked \$1000 of being as much as he had at the end of the first year, and the third year his gain lacked \$1000 of being as much as he had at the end of the second year. He found that at the end of the third year his capital was 3 times his original capital. What was his original capital?

82. A and B began business with equal capital. The first year A gained a sum equal to $\frac{1}{3}$ of his capital, and B lost $\frac{1}{4}$ of his. The second year A lost \$72 and B gained \$36, when it was found that B's capital was $\frac{3}{4}$ of A's. What was the original capital of each?

83. A cistern, which held 648 gallons of water, was filled in 18 minutes by two pipes, one of which conveyed 6 gallons more per minute than the other. How much did each convey per minute?

84. A farmer has 90 sheep in four fields. If the number in the first be increased by 2, the number in the second

diminished by 2, the number in the third multiplied by 2, and the number in the fourth divided by two, the results will be equal. How many are there in each flock?

85. A gentleman who had $10000, used a portion of it in building a house, and put the rest out at interest for one year: $\frac{1}{3}$ of it at 6% and $\frac{2}{3}$ of it at 5%. The income from both investments was $320. What was the cost of the house?

86. Paving a square court with stone at 40 cents a square yard will cost as much as inclosing it with a fence at a dollar per yard. What is the length of a side of the court?

87. Two soldiers start together for a fort. One, who travels 12 miles per day, after traveling 9 days, turns back as far as the other had traveled during those 9 days. He then turns and pursues his way toward the fort, where both arrive together 18 days from the time they set out. At what rate did the other travel?

88. A boy bought a certain number of apples at the rate of 4 for 5 cents, and sold them at the rate of 3 for 4 cents. He gained 60 cents. How many did he buy?

89. A gentleman left $315 to be divided among four servants, as follows: B was to receive as much as A and $\frac{1}{2}$ as much more; C was to receive as much as A and B and $\frac{1}{3}$ as much more; D was to receive as much as the other three and $\frac{1}{4}$ as much more. What was the share of each?

90. Two numbers are to each other as 2 to 3; but if 50 be subtracted from each, one will be $\frac{1}{2}$ the other. What are the numbers?

91. A woman sold eggs and apples. The eggs were worth 5 cents a dozen more than the apples; and 8 dozen eggs were worth as much as $13\frac{5}{7}$ dozen apples. What was the price of each per dozen?

92. Three men, A, B, and C, build 318 rods of wall. A builds 7 rods per day, B 6 rods, and C 5 rods. B

works twice as many days as A, and C works $\frac{1}{2}$ as many days as both A and B. How many days does each work?

93. A gentleman has two horses, and a carriage worth \$150. The value of the poorer horse and carriage is twice the value of the better horse; and the value of the better horse and carriage is three times the value of the poorer horse. What is the value of each horse?

94. A man bought two pieces of cloth, one of which lacked 12 yards of being 4 times as long as the other. The longer cost \$5 per yard, and the shorter \$4 per yard. Twenty-three yards being cut off from the longer, and 5 from the shorter, and each remainder being sold for a dollar a yard more than it cost, he received \$142. How many yards of each were there?

95. When, after 2 o'clock will the hour and minute hands of a clock be together?

SOLUTION.

Let $x =$ the number of minute-spaces that the minute hand travels before they come together.

Then, $\frac{x}{12} =$ the number of minute-spaces that the hour hand travels.

Then, since they were 10 minute-spaces apart at two o'clock,

$$x - \frac{x}{12} = 10$$

$$\frac{11x}{12} = 10$$

$$11x = 120$$

$$x = 10\tfrac{10}{11}, \text{ the number of minutes after 2}$$

96. When, after 5 o'clock, will the hour and minute hands of a clock be together?

97. When, after 8 o'clock, will the hour and minute hands of a clock be together?

98. When, after 4 o'clock, will the hour and minute hands of a clock make a straight line?

99. When, after 5 o'clock, will the hour and minute hands of a clock make a straight line?

100. When, first, after 6 o'clock, will the hour and minute hands of a clock be 15 minute-spaces apart?

101. When, after half-past 8 o'clock, will the hour and minute hands of a clock be 15 minute-spaces apart?

102. After paying out $\frac{1}{m}$ and $\frac{1}{n}$ of my money, I had b dollars left. How much had I at first?

SOLUTION.

Let $x =$ the amount I had at first.

Then, $\frac{x}{m} + \frac{x}{n} =$ the amount I spent.

$$\text{Therefore, } \frac{x}{m} + \frac{x}{n} + b = x$$

$$mx + nx + mnb = mnx$$

$$mnx - mx - nx = mnb$$

$$(mn - m - n)x = mnb$$

$$x = \frac{mnb}{mn - m - n}$$

184. A problem in which *literal notation* is used, is called a **General Problem.**

Such problems give an infinite number of numerical results, by assigning different numerical values to the literal quantities.

Thus, in problem 102, given above, when $m = 4$, $n = 5$, and $b = 66$, the value of x is 120; when $m = 5$, $n = 8$, and $b = 54$, the value of x is 80.

103. A horse and saddle are worth m dollars, and the horse is worth n times as much as the saddle. What is the value of each when $m = 200$ and $n = 9$?

104. A gentleman gave two servants b dollars, giving A a times as much as B. How much did he give each? How much did he give each if $b = 75$ and $a = 4$?

105. Divide the number b into two such parts that one shall be a times the other. What will be the result when $b = 24$ and $a = 7$?

106. If A can do a piece of work in n days and B in m days, in what time can both do it working together? What will be the result when n is 5 and m is 7? What when n is 10 and m 8?

107. A pleasure party of a persons hired a coach. If there had been b persons more, it would have cost each d dollars less than it did. How much did each one pay? What is the result when a is 8, b 4, and d \$1?

108. A certain number divided by b gives a result such that the sum of the dividend, divisor, and quotient is c. What is the number? What is the number when b is 16 and c is 84?

SIMULTANEOUS EQUATIONS.

TWO UNKNOWN QUANTITIES.

185. 1. When $x = 2$ and $y = 3$, what is the value of $x + y$? What of $2x + y$?

2. When $x = 4$ and $y = 3$, what is the value of $x + y$? Of $x - y$?

3. When $x = 6$ and $y = 2$, what is the value of $2x + y$? Of $x + 2y$?

4. When $x=10$ and $y=3$, what is the value of $2x+y$? Of $x+3y$?

5. Write down the results in each of the above in the form of equations. What is the value of x in each of the first two equations? Of y? Of x in each of the second two? Of y? Of x in the third two? Of y?

6. What are those equations called in which the same letter has the same value in each equation? (See Art. 185.)

7. What may be done to equations without destroying the equality? (See Axioms, Art. 59.)

8. If the members of the equation $x+y=4$ are multiplied by 2, what is the resulting equation? What is the resulting equation, when the equation $2x+y=8$ is multiplied by 3?

9. How can the equation $3x+6y=18$ be *derived* from $x+2y=6$? How can $4x+2y=8$ be derived from the equation $2x+y=4$?

10. If $x+y$ is added to $x-y$, what is the result? If $x+2y$ is added to $x-2y$, what is the result?

11. If the equation $x+y=8$ is added to the equation $x-y=4$, what is the resulting equation? How many unknown quantities does it contain? How many unknown quantities were there in the original equation?

12. If $x+y=8$ and $x+2y=12$, what equation will result by subtracting the first from the second? What is the value of y? What is the value of x?

13. If the sum of two numbers is 12, what are the numbers? How many answers may be given to the question?

14. In the equation $x+y=12$, how many values may x have? How many may y have?

15. What are those equations called in which the unknown quantities may have an infinite number of values? (See Art. 188.)

DEFINITIONS.

186. Simultaneous Equations are those in which the same unknown quantity has the same value in every equation.

Thus, $\left\{\begin{array}{l} x+y=12 \\ x-y=2 \end{array}\right\}$ are simultaneous equations in which $x=7$ and $y=5$.

187. Derived Equations are those which are obtained by combining other equations or performing some operation upon them.

Thus, $2x+2y=8$, is an equation *derived* from $x+y=4$, and $2x+3y=7$, is derived by adding $x+y=3$ and $x+2y=4$.

188. Independent Equations are such as can not be derived from one another or reduced to the same form.

Thus, $2x+y=5$ and $x+2y=6$, are independent equations.

189. An Indeterminate Equation is one in which the unknown quantities may have an infinite number of values.

Thus, $x+y=12$, is an indeterminate equation, because each of the unknown quantities may have an infinite number of values. Hence,

190. PRINCIPLES—1. *Every single equation containing two unknown quantities is indeterminate. Consequently,*

2. *In order to solve equations containing two unknown quantities, two independent equations, involving one or both of the quantities, must be given.*

191. Elimination is the process of deducing from simultaneous equations, equations containing a less number of unknown quantities than is found in the given equations.

CASE I.

192. Elimination by Addition and Subtraction.

1. When $x+y=8$ and $x-y=2$, how may the value of x be found?

2. When $x+2y=10$ and $x-2y=6$, how may the value of x be found?

3. When $3x+4y=16$ and $5x-4y=16$, how may the value of x be found?

4. When may a quantity be eliminated by addition?

5. When $x+2y=6$ and $x+y=4$, how may the value of y be found?

6. When $2x+3y=10$ and $x+3y=8$, how may the value of x be found?

7. When may a quantity be eliminated by subtraction?

193. PRINCIPLE.—*Quantities may be eliminated by addition or by subtraction when they have the same coëfficients.*

EXAMPLES.

1. Find the value of x and y in the equations $2x+3y=13$ and $3x+2y=12$.

PROCESS.

$$\begin{aligned}
2x+3y&=13 \qquad (1)\\
3x+2y&=12 \qquad (2)\\
6x+9y&=39 \qquad (3)\\
6x+4y&=24 \qquad (4)\\
5y&=15 \qquad (5)\\
y&=\ 3 \qquad (6)\\
2x+9\ &=13 \qquad (7)\\
2x&=\ 4 \qquad (8)\\
x&=\ 2 \qquad (9)
\end{aligned}$$

EXPLANATION.—Since the quantities in the given equations have not the same coëfficients, the first equation is multiplied by 3 and the second by 2, producing equations (3) and (4) in which the coëfficients of x are alike. Since the coëfficients of x are alike, and they have the same sign, x may be eliminated by subtraction (Prin.). Subtracting (4) from (3), we obtain (5). Dividing equation (5) by the coëfficient of y, we obtain (6).

Substituting the value of y in equation (1), the resulting equation is (7). Transposing and uniting, the value of $x = 2$.

RULE.—*If necessary, multiply or divide one or both equations so that one unknown quantity may have the same coëfficient in both.*

When the signs of the equal coëfficients are the same, subtract the equations; when the signs are unlike, add the equations.

Find the values of the unknown quantities in the following:

2. $\begin{cases} x + 2y = 7. \\ x + \ \ y = 5. \end{cases}$

3. $\begin{cases} 4x + 3y = 7. \\ 2x - 3y = -1. \end{cases}$

4. $\begin{cases} 4x - 5y = 3. \\ 3x + 5y = 11. \end{cases}$

5. $\begin{cases} 2x + 6y = 10. \\ 3x + 2y = 8. \end{cases}$

6. $\begin{cases} 8x + 3y = 22. \\ 4x + 5y = 18. \end{cases}$

7. $\begin{cases} 3x + 4y = 25. \\ 4x + 3y = 21. \end{cases}$

8. $\begin{cases} 5x + 6y = 61. \\ 4x + 5y = 50. \end{cases}$

9. $\begin{cases} 4x + 3y = 32. \\ 7x - 6y = 11. \end{cases}$

10 $\begin{cases} 5x + 6y = 40. \\ 8x - 4y = 4. \end{cases}$

11. $\begin{cases} 3x + 6y = 39. \\ 5x - 3y = 13. \end{cases}$

12. $\begin{cases} \dfrac{x}{2} + \dfrac{y}{3} = 3. \\ \dfrac{x}{5} + \dfrac{y}{2} = \dfrac{23}{10}. \end{cases}$

13. $\begin{cases} \dfrac{x}{6} - \dfrac{y}{3} = -\dfrac{1}{3}. \\ \dfrac{2x}{3} - \dfrac{3y}{4} = 1. \end{cases}$

14. $\begin{cases} \dfrac{5x}{6} + \dfrac{2y}{5} = 14. \\ \dfrac{3x}{4} - \dfrac{2y}{5} = 5. \end{cases}$

15. $\begin{cases} \frac{3x}{4} + \frac{2y}{3} = 12. \\ \frac{3x}{5} - \frac{y}{2} = \frac{3}{10}. \end{cases}$

16. $\begin{cases} \frac{4x}{5} + \frac{2y}{3} = 6. \\ \frac{2x}{3} + \frac{3y}{4} = 5\tfrac{7}{12}. \end{cases}$

17. $\begin{cases} \frac{3x}{5} + \frac{7y}{4} = 10. \\ \frac{2x}{7} - \frac{y}{5} = \frac{22}{35}. \end{cases}$

18. $\begin{cases} \frac{2x}{5} + \frac{3y}{4} = 13\tfrac{1}{2}. \\ \frac{5x}{6} + \frac{3y}{5} = 13\tfrac{1}{2}. \end{cases}$

CASE II.

194. Elimination by Comparison.

1. If, in the equation $x + y = 8$, y is transposed to the second member, what will be the form of the equation?

2. If, in the equation $x - y = 4$, y is transposed to the second member, what will be the form of the equation?

3. If, in the simultaneous equations $x + 2y = 8$ and $x - y = 5$, y in each is transposed to the second member, what will be the form of the equations?

4. Since the second members of these derived equations are each equal to x, how will they compare with each other?

5. If these second members are formed into an equation, how many unknown quantities will it contain?

6. How may an unknown quantity be eliminated from two simultaneous equations by *comparison*?

EXAMPLES.

1. Find the value of x and of y in the equations $x + 2y = 8$ and $3x + 2y = 12$.

PROCESS.

$$\begin{aligned}
x + 2y &= 8 && (1)\\
3x + 2y &= 12 && (2)\\
\hline
x &= 8 - 2y && (3)\\
x &= \frac{12 - 2y}{3} && (4)\\
\frac{12 - 2y}{3} &= 8 - 2y && (5)\\
12 - 2y &= 24 - 6y && (6)\\
4y &= 12 && (7)\\
y &= 3 && (8)\\
x + 6 &= 8 && (9)\\
x &= 2 && (10)
\end{aligned}$$

EXPLANATION.—Since, in elimination by comparison, the value of the same unknown quantity in each equation is to be found, and a new equation is to be formed from them, $2y$ in equation (1) is transposed, giving (3). Transposing $2y$ in (2) and dividing by 3, equation (4) is obtained. Since these two values of x are equal, equation (5) is obtained. Clearing of fractions, we obtain (6). Transposing and uniting, (7) is obtained. Dividing by 4, we obtain (8). Substituting this value of y in equation (1), we obtain (9). Uniting, we obtain (10).

RULE.—*Find an expression for the value of the same unknown quantity in each equation.*

Place these values equal to each other, and solve the equation.

Solve the following equations by comparison:

2. $\begin{cases} 3x + y = 9. \\ x + 2y = 8. \end{cases}$

3. $\begin{cases} 2x - y = 3. \\ x + 3y = 19. \end{cases}$

4. $\begin{cases} 4x + 2y = 26. \\ 3x + 4y = 39. \end{cases}$

5. $\begin{cases} 2x - 3y = -14. \\ 3x + 2y = 44. \end{cases}$

6. $\begin{cases} 3x + 4y = 18. \\ x + 2y = 8. \end{cases}$

7. $\begin{cases} x + 6y = 13. \\ 5x + 2y = 9. \end{cases}$

8. $\begin{cases} 4x + 2y = 26. \\ 3x - 4y = 3. \end{cases}$

9. $\begin{cases} 2x - 3y = -7. \\ 4x - 5y = -9. \end{cases}$

10. $\begin{cases} 3x + 2y = 33. \\ 9x - 4y = 9. \end{cases}$

11. $\begin{cases} 6x + y = 45. \\ 3x - 2y = 15. \end{cases}$

12. $\begin{cases} 4x - 5y = -34. \\ 2x - 3y = -22. \end{cases}$

13. $\begin{cases} x + 4y = 11. \\ 5x - 2y = 11. \end{cases}$

14. $\begin{cases} 2x - 3y = 3. \\ 4x + 5y = 39. \end{cases}$

15. $\begin{cases} \dfrac{x}{2} + \dfrac{y}{3} = 5. \\[2ex] \dfrac{x}{3} + \dfrac{y}{2} = 5. \end{cases}$

16. $\begin{cases} \dfrac{3x}{5} + \dfrac{2y}{3} = 17. \\[2ex] \dfrac{2x}{3} + \dfrac{3y}{4} = 19. \end{cases}$

17. $\begin{cases} \dfrac{2x}{7} + \dfrac{2y}{3} = 5\frac{1}{3}. \\[2ex] \dfrac{3x}{5} + \dfrac{3y}{5} = 7\frac{1}{5}. \end{cases}$

18. $\begin{cases} \dfrac{3x}{5} - \dfrac{y}{4} = 4\frac{3}{4}. \\[2ex] \dfrac{2x}{7} - \dfrac{2y}{5} = 3\frac{5}{14}. \end{cases}$

19. $\begin{cases} \dfrac{4x}{5} + \dfrac{2y}{3} = 24\frac{4}{5}. \\[2ex] \dfrac{3x}{7} - \dfrac{5y}{6} = -9. \end{cases}$

CASE III.

195. Elimination by Substitution.

1. In the equation $x + y = 5$, if $x = 2$, what is the value of y? How is this value obtained?

2. In an equation containing two unknown quantities, if the value of one quantity is given, how may the value of the other be found?

3. If y is transposed to the second member in the equation $x + y = 5$, what will be the expression for the value of x?

4. If x is transposed to the second member in the equation $x + y = 6$, what will be the expression for the value of y?

5. Express the value of x in the first of the simultaneous equations $x + y = 5$ and $x + 2y = 7$. When the value of x is obtained, how may the value of y be obtained?

6. How may an unknown quantity be eliminated from simultaneous equations by *substitution*?

EXAMPLES.

1. Find the value of x and of y in the equations $3x + 2y = 12$ and $2x + 3y = 13$.

PROCESS.

$$3x + 2y = 12 \qquad (1)$$
$$2x + 3y = 13 \qquad (2)$$
$$x = \frac{12 - 2y}{3} \qquad (3)$$
$$\frac{24 - 4y}{3} + 3y = 13 \qquad (4)$$
$$24 - 4y + 9y = 39 \qquad (5)$$
$$5y = 15 \qquad (6)$$
$$y = 3 \qquad (7)$$
$$x = \frac{12 - 6}{3} = 2 \qquad (8)$$

EXPLANATION.—Since one unknown quantity can be eliminated by finding its value in one of the given equations and substituting this value in another, we find the value of x from (1) and obtain (3). Substituting this value in (2), (4) is obtained. Clearing of fractions, the resulting equation is (5). Uniting terms we obtain (6). Dividing, $y = 3$. Substituting this value in (3), $x = 2$.

RULE.—*Find an expression for the value of one of the unknown quantities in one of the equations.*

Substitute this value for the same unknown quantity in the other equation, and solve the equation.

Solve the following by substitution:

2. $\begin{cases} x + 2y = 10. \\ 2x - 3y = -1. \end{cases}$

3. $\begin{cases} 3x - 2y = 1. \\ x + 4y = 19. \end{cases}$

4. $\begin{cases} x - 2y = 6. \\ 2x - y = 27. \end{cases}$

5. $\begin{cases} 9x - y = 6. \\ x + y = 4. \end{cases}$

6. $\begin{cases} 3x + 5y = 2. \\ 6x + 5y = 3. \end{cases}$

7. $\begin{cases} 7x - 5y = 13. \\ 3x + 3y = 21. \end{cases}$

8. $\begin{cases} 6x + y = 60. \\ 3x + 2y = 39. \end{cases}$

9. $\begin{cases} 2x + 5y = 29. \\ 2x - 5y = -21. \end{cases}$

10. $\begin{cases} \dfrac{x}{5} + \dfrac{y}{6} = 18. \\ \dfrac{x}{2} - \dfrac{y}{4} = 21. \end{cases}$

11. $\begin{cases} x + 5y = 41. \\ 3x - 2y = 21. \end{cases}$

12. $\begin{cases} \dfrac{x}{2} + \dfrac{y}{3} = 7. \\ \dfrac{x}{3} + \dfrac{y}{4} = 5. \end{cases}$

13. $\begin{cases} \dfrac{x}{3} + \dfrac{z}{4} = 8. \\ x - z = -3. \end{cases}$

14. $\begin{cases} \dfrac{x+y}{3} = 5. \\ \dfrac{x-y}{2} = 2\tfrac{1}{2}. \end{cases}$

15. $\begin{cases} \dfrac{x}{7} + 7y = 251. \\ \dfrac{y}{7} + 7x = 299. \end{cases}$

Solve the following by any method:

16. $\begin{cases} \dfrac{1}{x} + \dfrac{2}{y} = 10. \\ \dfrac{4}{x} + \dfrac{3}{y} = 20. \end{cases}$

17. $\begin{cases} \dfrac{7}{x} + \dfrac{5}{y} = 19. \\ \dfrac{8}{x} - \dfrac{3}{y} = 7. \end{cases}$

18. $\begin{cases} \dfrac{5}{3x} + \dfrac{2}{5y} = 7. \\ \dfrac{7}{6x} - \dfrac{1}{10y} = 3. \end{cases}$

19. $\begin{cases} \dfrac{a}{x} + \dfrac{b}{y} = m. \\ \dfrac{a}{x} - \dfrac{b}{y} = n. \end{cases}$

20. Given $\left\{\begin{aligned} &\frac{x}{4}+8=\frac{y}{2}-12 \\ &\frac{x+y}{5}+\frac{y}{3}=\frac{2x-y}{4}+35 \end{aligned}\right\}$, to find x and y.

21. Given $\left\{\begin{aligned} &\frac{2x-y}{7}+3x=2y-6 \\ &\frac{y+3}{5}+\frac{y-x}{6}=2x-8 \end{aligned}\right\}$, to find x and y.

22. Given $\left\{\begin{aligned} &\frac{x-2}{5}-\frac{10-x}{3}=\frac{y-10}{4} \\ &\frac{2y+4}{3}=\frac{4x+y+13}{8} \end{aligned}\right\}$, to find x and y

23. Given $\left\{\begin{aligned} &\frac{x+1}{y-1}-\frac{x-1}{y}=\frac{6}{7} \\ &x-y=1 \end{aligned}\right\}$, to find x and y.

24. Given $\left\{\begin{aligned} &\frac{1-3x}{7}+\frac{3y-1}{5}=2 \\ &\frac{3x+y}{11}+y=9 \end{aligned}\right\}$, to find x and y.

25. Given $\left\{\begin{aligned} &4x+y=11 \\ &\frac{y}{5x}=\frac{7x-y}{3x}-\frac{23}{15} \end{aligned}\right\}$, to find x and y.

26. Given $\left\{\begin{aligned} &\frac{x}{a}+\frac{y}{b}=2. \\ &bx-ay=0. \end{aligned}\right\}$, to find x and y.

27. $\left\{\begin{array}{l}\dfrac{x}{c}+\dfrac{y}{c}=1. \\ \dfrac{ax-by}{a-b}=c.\end{array}\right\}$

28. $\left\{\begin{array}{l}\dfrac{a}{b+y}=\dfrac{b}{3a+x}. \\ ax+2by=d.\end{array}\right\}$

29. $\left\{\begin{array}{l}\dfrac{x}{m}+\dfrac{y}{n}=2. \\ \dfrac{x}{m}-\dfrac{y}{n}=1.\end{array}\right\}$

30. $\left\{\begin{array}{l}abx+cdy=2. \\ ax-cy=\dfrac{d-b}{bd}.\end{array}\right\}$

PROBLEMS.

196. 1. If 7 lb. of tea and 5 lb. of coffee cost \$5.50, and 6 lb. of tea and 3 lb. of coffee cost \$4.20, what was the price per pound of each?

PROCESS.

Let $x=$ the price of tea per lb.
Let $y=$ the price of coffee per lb.

$$\begin{array}{rl} 7x+5y=\$5.50 & (1) \\ 6x+3y=\$4.20 & (2) \\ \hline x=\$\ .50 & (3) \\ y=\$\ .40 & (4) \end{array}$$

EXPLANATION.—Since there are two kinds of quantities involved, namely, tea and coffee, x may be used to represent the price per pound of the tea, and y the price per pound of the coffee. Then from the conditions of the problem we have equations (1) and (2). Solving them, the value of x is \$.50 and y is \$.40.

2. There is a fraction such that if 1 is added to the numerator the value of the fraction will be 1; and if 3 is added to the denominator the value will be $\frac{1}{2}$. What is the fraction?

SOLUTION.

Let $x =$ the numerator.

$y =$ the denominator.

Then, $\frac{x}{y} =$ the fraction.

$$\frac{x+1}{y} = 1 \qquad (1)$$

$$\frac{x}{y+3} = \tfrac{1}{2} \qquad (2)$$

$$x = 4 \qquad (3)$$

$$y = 5 \qquad (4)$$

$$\frac{x}{y} = \frac{4}{5} \qquad (5)$$

3. There is a number such that if it be divided by the sum of the digits which express it the quotient will be 4, and if 36 be added to it the sum will be expressed by the digits inverted. What is the number?

SOLUTION.

Let $x =$ the digit in tens' place.

$y =$ the digit in units' place.

$10x + y =$ the number.

$10y + x =$ the number when the digits are inverted.

$$\frac{10x+y}{x+y} = 4 \qquad (1)$$

$$10x + y + 36 = 10y + x \qquad (2)$$

$$x = 4 \qquad (3)$$

$$y = 8 \qquad (4)$$

$$10x + y = 48 \qquad (5)$$

4. The sum of two numbers is 24, and their difference is 8. What are the numbers?

5. The sum of two numbers is 29, and their difference is 5. What are the numbers?

6. The sum of two numbers divided by 2 gives a quotient of 24, and their difference divided by 2 gives a quotient of 17. What are the numbers?

7. A man hired for one day 6 men and 2 boys for \$28, and afterward, at the same rate, 3 men and 4 boys for \$20. What was paid each per day?

8. There is a fraction such that if 3 be added to the numerator its value will be $\frac{1}{3}$, and if 1 be subtracted from the denominator its value will be $\frac{1}{5}$. What is the fraction?

9. A man has two horses, and a saddle worth \$10. The value of the saddle and the first horse is double that of the second horse, but the value of the saddle and the second horse lacks \$13 of being equal to the value of the first horse. What is the value of each horse?

10. Two purses contain together \$300. If \$30 is taken from the first and put into the second, there will be the same amount in each. How much money is there in each?

11. A and B have \$570. If A's money were three times, and B's were five times as great as it really is, they would have \$2350. How much has each?

12. What fraction is that to the numerator of which if 4 be added, the value will be $\frac{1}{2}$, and if 7 be added to the denominator the value will be $\frac{1}{5}$?

13. There is a number of two digits, which is equal to 4 times the sum of the digits, and if 18 be added to the number, the result will be expressed by the digits inverted. What is the number?

14. A person had two kinds of money, such that it took 10 pieces of one kind to make a dollar, and two pieces of the other to make a dollar. He paid a man a dollar, giving him 6 pieces. How many of each kind were used?

15. A party which had hired a coach, found that if there had been three more persons, they would each have had to pay \$1 less than they did; and if there had been 2 less they would each have had to pay \$1 more. How many persons were there? How much did each pay?

16. A wine-merchant sold at one time 20 dozen of port wine and 30 dozen of sherry for £120. At another time he sold 30 dozen of port and 25 dozen of sherry for £140. What was the price per dozen of each?

17. There is a number expressed by two figures. If to the sum of the digits 7 is added, the result will be 3 times the left-hand digit, and if 18 is subtracted from the number, the digits will be inverted. What is the number?

18. A and B had together a capital of \$9800. A invested $\frac{1}{8}$ of his capital and B $\frac{1}{6}$ of his, when each had the same sum left. How much had each before the investment?

19. A farmer purchased 100 acres of land for \$2450. For a part of it he paid \$20 an acre and for the rest \$30 an acre. How many acres were there in each part?

20. The sum of the ages of a father and a son is 80 years. If the age of the son is doubled, it will exceed the age of the father by 10 years. What is the age of each?

21. A said to B: "Give me 20 cents of your money and I will have 4 times as much as you." B said to A: "Give me 20 cents of your money and I will have $1\frac{1}{2}$ times as much as you." How much had each?

22. A farmer bought 100 acres of land, part at \$37 and part at \$45 an acre, paying for the whole \$4220. How much land was there in each part?

23. A boy expended 30 cents for apples and pears, buying the apples at 4 for a cent and the pears at 5 for a cent.

He then sold $\frac{1}{2}$ of his apples and $\frac{1}{3}$ of his pears for 13 cents, which was what they cost him. How many of each did he buy?

24. A railway train, after traveling an hour, is detained 30 minutes. It then proceeds at $\frac{3}{5}$ of its former rate, and arrives 10 minutes late. If the detention had occurred 12 miles further on the train would have arrived 4 minutes later than it did. At what rate did the train travel before the detention, and what was the whole distance traveled?

THREE OR MORE UNKNOWN QUANTITIES.

197. 1. In the equations $x + 2y + z = 8$ and $2x + 3y + 2z = 14$, how may x be eliminated?

2. In the equations $2x + 3y + 4z = 26$ and $x + 4y + 2z = 18$, how may z be eliminated?

3. If one of the quantities in the above equations is eliminated, how many quantities will be left?

4. How many independent equations are necessary before the values of two unknown quantities can be found?

5. How many independent equations containing the same two unknown quantities can be formed from the equations in (1)? From the equations in (2)?

6. Since there must be two independent equations given so that the values of two unknown quantities may be found, and since from the two equations given in (1) and (2) only one derived equation can be formed, how many independent equations must be given so that the value of any of the unknown quantities may be found?

7. When the values of two unknown quantities are known, how may the value of a third be found from an equation containing three unknown quantities?

198. Since it is necessary to have *two* independent equations to find the values of *two* unknown quantities, and *three* independent equations to find the values of *three* unknown quantities, etc., a general law may be expressed as follows:

PRINCIPLE.—*To find the values of unknown quantities, there must be as many independent equations as there are unknown quantities.*

EXAMPLES.

1. Given $\left\{\begin{array}{r} x+2y+3z=14 \\ 2x+\ y+2z=10 \\ 3x+4y-3z=\ 2 \end{array}\right\}$, to find x, y, and z.

PROCESS.

$$
\begin{array}{rl}
x+2y+3z=14 & (1) \\
2x+y+2z=10 & (2) \\
3x+4y-3z=2 & (3) \\
\hline
2x+4y+6z=28 & (4) \\
2x+y+2z=10 & \\
\hline
3y+4z=18 & (5) \\
3x+6y+9z=42 & (6) \\
3x+4y-3z=2 & \\
\hline
2y+12z=40 & (7) \\
9y+12z=54 & (8) \\
\hline
7y=14 & (9) \\
y=2 & (10) \\
4+12z=40 & (11) \\
12z=36 & (12) \\
z=3 & (13) \\
x+4+9=14 & (14) \\
x=1 & (15)
\end{array}
$$

EXPLANATION.—To eliminate x from the first two equations, we multiply (1) by 2 and obtain equation (4). Subtracting equation (2) from (4), we have (5). Eliminating x by a similar process from (1) and (3), the resulting equation is (7). From (5) and (7) we eliminate z and obtain (9), which contains only y. Equation (10) gives us the value of y. Substituting this value in equation (7), we obtain (11), and the value of z is found in (13). Substituting the values of y and z in (1), we have equation (14), from which we find the value of x to be 1.

RULE.—*Combine the equations so as to eliminate the same unknown quantity from each, obtaining a set of derived equations containing one less unknown quantity.*

Combine these derived equations so as to eliminate a second unknown quantity, and thus continue until an equation is found containing but one unknown quantity. Then find the value of this unknown quantity.

Substitute this value in one of the equations containing two unknown quantities, and obtain the value of a second quantity.

Substitute the two values already found in an equation containing three unknown quantities, and find the value of a third quantity, and thus continue until the values of all the unknown quantities are found.

Find the value of each unknown quantity in the following:

2. $$\left\{\begin{array}{r} x-2y+2z=5. \\ 5x+3y+6z=57. \\ x+2y+2z=21. \end{array}\right\}$$

3. $$\left\{\begin{array}{r} 7x-4y+3z=35. \\ 4x-5y+2z=6. \\ 2x+3y-z=20. \end{array}\right\}$$

4. $$\left\{\begin{array}{r} x+y+z=6. \\ 5x+4y+3z=22. \\ 3x+4y-3z=2. \end{array}\right\}$$

5. $$\left\{\begin{array}{r} x-4y+3z=2. \\ 4x-3y+z=9. \\ 2x+6y-4z=14. \end{array}\right\}$$

6. $$\left\{\begin{array}{r} x+y+z=35. \\ x-2y+3z=15. \\ y-x+z=-5. \end{array}\right\}$$

7. $$\left\{\begin{array}{r} x+22=y+z. \\ y+22=2x+2z. \\ z+22=3x+3y. \end{array}\right\}$$

8. $$\left\{\begin{array}{r} x+y+z=12. \\ x-y=2. \\ x-z=4. \end{array}\right\}$$

9. $$\left\{\begin{array}{r} u+y+z=2x. \\ u+x+z=3y. \\ u+x+y=4z. \\ u+x=y+36. \end{array}\right\}$$

10. $\begin{cases} x + y + 2z + w = 18. \\ x + 2y + z + w = 17. \\ x + y + z + 2w = 19. \\ 2x + y + z + w = 16. \end{cases}$

11. $\begin{cases} u + v + x + y = 14. \\ u + v + x + z = 15. \\ u + v + y + z = 16. \\ u + x + y + z = 17. \\ v + x + y + z = 18. \end{cases}$

By studying the equations a little before commencing the solution, the student will often discover modes of solution that will simplify the work very much.

Thus, in example 12 the quantities may be eliminated *without clearing* of fractions.

In example 8, by finding the sum of the three equations, the value of x may be found at once.

In example 10, by finding the sum of the four given equations, and dividing by 5, the value of the sum of the unknown quantities is found. This, subtracted from each of the given equations successively, gives the values of the unknown quantities.

Find the value of each unknown quantity in the following:

12. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = 5. \\ \dfrac{1}{y} + \dfrac{1}{z} = 7. \\ \dfrac{1}{x} + \dfrac{1}{z} = 6. \end{cases}$

13. $\begin{cases} x + \dfrac{1}{3}y = 5. \\ x + \dfrac{1}{3}z = 6. \\ y + \dfrac{1}{3}z = 9. \end{cases}$

14. $\begin{cases} x+y=a. \\ x+z=b. \\ y+z=c. \end{cases}$

15. $\begin{cases} x+y=9. \\ y+z=11. \\ z+w=13. \\ w+u=15. \\ u+x=12. \end{cases}$

16. $\begin{cases} \dfrac{x}{a}+\dfrac{y}{b}=2. \\ \dfrac{x}{a}+\dfrac{z}{c}=2. \\ \dfrac{y}{b}+\dfrac{z}{c}=2. \end{cases}$

17. $\begin{cases} 3x+4y+z=35. \\ 3z+2y-3t=4. \\ 2x-y+2t=17. \\ 3z-2t+u=9. \\ t+y=13. \end{cases}$

18. $\begin{cases} \dfrac{xy}{x+y}=\dfrac{1}{5}. \\ \dfrac{yz}{x+z}=\dfrac{1}{6}. \\ \dfrac{xz}{x+z}=\dfrac{1}{7}. \end{cases}$

PROBLEMS.

199. 1. Find three numbers such that their sum is 60; $\frac{1}{2}$ of the first plus $\frac{1}{3}$ of the second, and $\frac{1}{5}$ of the third is 19; and twice the first with three times the remainder, when the third is subtracted from the second, is 50.

2. Find three numbers such that the first with $\frac{1}{2}$ of the sum of the second and third is 119; the second with $\frac{1}{3}$ of the remainder, when the first is subtracted from the third, is 68; and $\frac{1}{2}$ the sum of the three numbers is 94.

3. A, B, and C together possess \$1500. If B gives A \$200 of his money, A will have \$280 more than B; but if B should receive \$180 from C, B and C would have equal amounts. How much has each?

4. Three persons purchased sugar, coffee, and tea at the same rates. A paid \$4.20 for 7 pounds of sugar, 5 pounds of coffee, and 3 pounds of tea; B paid \$3.40 for 9 pounds of sugar, 4 pounds of coffee, and 2 pounds of tea; C paid \$3.25 for 5 pounds of sugar, 2 pounds of coffee, and 3 pounds of tea. What was the price of each per pound?

5. Divide 125 into four such parts that, if the first is increased by 4, the second diminished by 4, the third multiplied by 4, and the fourth divided by 4, the sum, product, difference, and quotient shall all be equal.

6. A and B can perform a piece of work in 8 days; A and C can do it in 9 days, and B and C in 10 days. In how many days can each do the same work alone?

7. A certain number is expressed by three digits whose sum is 10. The sum of the first and last digits is $\frac{2}{3}$ of the second digit; and, if 198 be subtracted from the number, the digits will be inverted. What is the number?

Let $x =$ the first digit, or hundreds; y, the second digit, or tens; z, the third digit, or units. Then, $100x + 10y + z =$ the number.

8. There are two fractions which have the same denominator. If 1 be subtracted from the numerator of the smaller, its value will be $\frac{1}{3}$ of the larger fraction; but if 1 be subtracted from the numerator of the larger, its value will be twice that of the smaller. The difference between the fractions is $\frac{1}{3}$. What are the fractions?

9. A man divided a sum of money among his four sons, so that the share of the eldest was $\frac{1}{2}$ of the shares of the other three; the share of the second $\frac{1}{3}$ of the shares of the other three, and the share of the third $\frac{1}{4}$ of the shares of the other three. The eldest had \$14 more than the youngest. What was the share of each?

10. A farmer found that the number of his sheep was 26 more than the number of his cows and horses together; that $\frac{1}{5}$ of the number of sheep was equal to the number of horses together with $\frac{1}{4}$ of his cows; and that $\frac{1}{8}$ of his cows, $\frac{1}{2}$ of his horses, and $\frac{1}{5}$ of his sheep amounted to 12. How many had he of each?

11. There are three purses such that if \$20 is taken out of the first and put into the second, it will contain four times as much as remains in the first; if \$60 is taken from the second and put into the third, the third will contain $1\frac{3}{4}$ times as much as remains in the second; if \$40 is taken from the third and put into the first, the third will contain $2\frac{1}{4}$ times as much as the first. How much is there in each purse?

ZERO AND INFINITY.

200. How much is 2 times 0? 3 times 0? 500 times 0? a times 0? Any number of times 0?

PRINCIPLE 1.—*When zero is multiplied by a finite quantity the product is zero.*

201. How much is 0 divided by 2? 0 divided by 6? 0 divided by a? 0 divided by any number?

PRINCIPLE 2.—*When zero is divided by any finite quantity the quotient is zero.*

202. 1. Since 2 times $0=0$, 3 times $0=0$, 500 times $0=0$, and a times $0=0$, if both members of each equation are divided by 0, one of the factors, what will be the results?

2. Since the value of $\frac{0}{0}$ is found to be equal to 2, 3,

500, and a, what may be said of the value of the expression $\frac{0}{0}$?

PRINCIPLE 3.—*When zero is divided by zero the quotient may be any finite quantity, or, it is indeterminate.*

203. 1. What is the quotient of 2 divided by $\frac{1}{2}$? By $\frac{1}{4}$? By $\frac{1}{8}$? By $\frac{1}{16}$?

2. When the divisor is diminished, while the dividend remains the same, what effect is produced upon the quotient?

3. If the divisor is made *very small*, what will be the effect upon the quotient? What, when the divisor becomes *infinitely* small or *zero*

PRINCIPLE 4.—*When a finite quantity is divided by zero the quotient is infinitely large.*

204. 1. What is the quotient when 4 is divided by 2? By 4? By 8? By 16?

2. When the divisor is increased, the dividend remaining the same, what is the effect upon the quotient? What, when the divisor becomes *infinitely* large?

PRINCIPLE 5.—*When a finite quantity is divided by an infinitely large quantity the quotient is zero.*

205. The preceding principles may be expressed by algebraic formulas as follows,—the sign (∞) being used to indicate infinity:

Principle 1, $0 \times a = 0$.

Principle 2, $\dfrac{0}{a} = 0$.

Principle 3, $\dfrac{0}{0} =$ indeterminate result.

Principle 4, $\dfrac{a}{0} = \infty$.

Principle 5, $\dfrac{a}{\infty} = 0$.

EXAMPLES.

206. 1. What number is that whose third part exceeds its fourth part by as much as $\frac{29}{60}$ of it exceeds $\frac{2}{5}$ of it?

PROCESS.

Let $x =$ the number.

$$\frac{x}{3} - \frac{x}{4} = \frac{29x}{60} - \frac{2x}{5}$$

$$20x - 15x = 29x - 24x$$

$$44x - 44x = 0$$

$$(44 - 44)x = 0$$

$$x = \frac{0}{44 - 44} = \frac{0}{0}$$

EXPLANATION.—Solving the example as shown, the value of x is found to be $\frac{0}{0}$; that is, it is indeterminate.

This result may be interpreted to mean that *every* number will fulfill the conditions of the problem.

2. Find a number such that when 5 is added to 3 times the number, and the result is divided by the number increased by 2, the quotient will be 3.

The solution of this example gives the number to be ∞; that is, there is no finite number which will fulfill the conditions, and consequently the problem is *impossible*.

3. What number is there such that when $\frac{5}{8}$ of it is diminished by 4, the result is 3 less than $\frac{1}{2}$ of it plus $\frac{1}{8}$ of it?

4. I bought 400 sheep in two flocks, paying \$1.50 per head for the first flock and \$2 for the second. I lost 30 of the first and 56 of the second, and sold the rest of the first at \$2 per head and the second for \$2.50 without gain or loss. Required the number of each flock.

GENERAL PROBLEMS.

207. 1. The sum of two numbers is a and their difference is b. What are the numbers?

SOLUTION.

Let $x =$ the greater.

Let $y =$ the less.

$$x + y = a \qquad (1)$$
$$x - y = b \qquad (2)$$
$$2x = a + b \qquad (3)$$
$$x = \frac{a + b}{2} \qquad (4)$$
$$y = \frac{a - b}{2} \qquad (5)$$

The general rule for the solution of problems when the sum and the difference of two quantities are given, may be derived from the values of x and y obtained above. It is as follows:

The greater is equal to one-half their sum and difference. The less is equal to one-half the remainder when their difference is subtracted from their sum.

2. A can do a piece of work in a days and B can do it in b days. In what time can both do it working together? Write a general rule for the solution of problems like this. What will be the time if $a = 10$ and $b = 12$?

3. A is a times as old as B, and in b years he will be n times as old. What is the age of each? Write a general rule for the solution of problems like this. What will be the ages if $a = 6$, $b = 3$, and $n = 4$?

4. A traveler sets out from a place traveling a miles per day; after n days another follows him at the rate of b miles per day. In how many days will the second overtake the first? Write a general rule from the values of the unknown quantities.

INVOLUTION.

208. 1. What is the second power of a? Of b? Of c? Of d?

2. What is the third power of a? Of b? Of c? Of d?

3. How many times is a used as a factor in producing a^2? a^3? a^4? a^5? a^6? a^n?

4. How many times is a quantity used as a factor in producing the *second* power? *Third* power? *Fourth* power? *Fifth* power? The nth power?

5. What sign has the second power of $+a$? The third power? The fourth power? The fifth power? The sixth power?

6. What sign has the second power of $-a$? The third power? The fourth power?

7. Which powers of a negative quantity are *positive*? Which *negative*?

DEFINITIONS.

209. Involution is the process of finding the power of a quantity.

210. Power (Art. 18). **Exponent** (Art. 17). **Names of Powers** (Art. 19).

211. PRINCIPLES.—1. *All powers of a positive quantity are positive.*

2. *All even powers of a negative quantity are positive, and all odd powers are negative.*

CASE I.

212. Involution of monomials.

1. What is the third power of $6a^2b$?

PROCESS.

$$(6a^2b)^3 = 6a^2b \times 6a^2b \times 6a^2b =$$
$$6 \times 6 \times 6a^2a^2a^2bbb =$$
$$216a^6b^3$$

EXPLANATION.—Since in finding the third power of a quantity the quantity is used three times as a factor, *each factor* of the given quantity is used three times as a factor.

The product of the factors is $216a^6b^3$, the third power of the quantity.

RULE.—*Raise the numerical coëfficient to the required power, and multiply the exponent of each literal quantity by the exponent of the power to which it is to be raised, prefixing the proper sign to the result.*

2. Find the square of $6x^2y$.
3. Find the cube of $-4a^2b^2$.
4. Find the third power of $-3ab^3$.
5. Find the square of $-3c^2d^2$.
6. Find the fifth power of $2ax^3y^2$.
7. Find the sixth power of $2x^2yz^3$.
8. Find the fourth power of $4ab^4d^3$.
9. Find the seventh power of $-a^5b^2c^6$.
10. Find the fourth power of $-4a^2b^2c^4$.
11. Find the fifth power of $2x^2yz^4$.
12. Find the third power of $-5a^3bc^2$.
13. Find the eighth power of $a^4b^{-2}c^{-3}$.
14. Find the fourth power of $-4a^2b^3c^{-4}$.
15. Find the ninth power of $2a^{-2}b^2c^{\frac{1}{3}}$.

Raise to the required power

16. $(2x^2y^3z)^4$.
17. $(3x^{-2}y^{-4})^3$.
18. $(-4a^3z^2y)^3$.
19. $(-2a^{-2}y^{-3})^5$.
20. $(2ax^2y^{-3})^4$.
21. $(x^{-4}y^{-2}z^n)^3$.

Raise to the required power

22. $(x^my^{-n}z^{-m})^5$.
23. $(a^{-2n}z^nw^m)^4$.
24. $(a^{-4}y^{-5n}z^{-3n})^2$.
25. $(-x^4y^{-3}z^{-n})^n$.
26. $(-a^4b^2c^{-n}d^{-2n})^{5n}$.
27. $(a^3b^{-4}c^3d^{-n-2})^{n-2}$.

28. What is the third power of $\dfrac{2x^2y}{3a^2b}$?

PROCESS.

$$\left(\frac{2x^2y}{3a^2b}\right)^3 = \frac{2x^2y}{3a^2b} \times \frac{2x^2y}{3a^2b} \times \frac{2x^2y}{3a^2b} = \frac{8x^6y^3}{27a^6b^3}$$

EXPLANATION.—In raising a fraction to a power, both numerator and denominator must be raised to the required power.

29. Find the square of $\dfrac{2a}{3b}$.

30. Find the square of $\dfrac{2x^2}{3y}$.

31. Find the cube of $-\dfrac{6xy}{5ab}$.

32. Find the fourth power of $\dfrac{8a^2b}{7x^2y}$.

33. Find the sixth power of $\dfrac{a^{-2}b^n}{c^{-4}d^3}$.

34. Find the seventh power of $\dfrac{x^{-n}y^{-2n}}{a^nz^4}$.

35. Find the *n*th power of $\dfrac{a^2b^2c^{n-1}}{x^{n-2}y^{n-3}}$.

36. Find the 2*n*th power of $\dfrac{a^2b^3c^{-4}}{x^{-4}y^{-4}z^4}$.

CASE II.

213. Involution of polynomials.

EXAMPLES.

1. Find the third power of $x+y$.

The process is simply to use $x+y$ as a factor three times.

RULE.—*Use the given quantity as a factor as many times as there are units in the exponent of the required power.*

2. Find the second power of $(a+b)$.
3. Find the second power of $(a+1)$.
4. Find the third power of $(x+y)$.
5. Find the third power of $(2+x)$.
6. Find the third power of $(3+y)$.
7. Find the second power of $(2+b^2)$.
8. Find the second power of (a^2-b).
9. Find the second power of $(a+b+c)$.
10. Find the fourth power of $(x+2y)$.
11. Find the fourth power of $(n-m)$.
12. Find the third power of $(a+b)$.
13. Find the second power of (x^n+y^n).
14. Find the third power of $(2a+3b)$.
15. Find the third power of $(3x-2y)$.
16. Find the second power of (n^2+m^2).
17. Find the fifth power of $(a+b)$.
18. Find the sixth power of $(x+y)$.
19. Find the fourth power of $(2a+2b)$.
20. Find the seventh power of $(a+x)$.
21. Find the fifth power of $(3x+2z)$.
22. Find the fourth power of $(2y+z^2)$.
23. Find the square of $(a+b+c+d)$.

CASE III.

214. Special method of squaring a polynomial.

1. Examining the result obtained in the solution of the last example, which was *squaring* $(a + b + c + d)$, how many terms are squares? Of what are they the squares?

2. What is the coëfficient of each of the other terms?

3. How many terms contain a? By what quantities is a multiplied in the terms which contain it?

4. What other terms contain b? By what is b multiplied in the terms which contain it?

5. How many other terms contain c? By what is c multiplied in the terms which contain it?

215. PRINCIPLE.—*The square of a polynomial contains the square of each term and twice the product of each term multiplied by each of the terms following it.*

EXAMPLES.

1. Square $(a + b + c + d)$.

PROCESS.

$$(a + b + c + d)^2 = a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd$$

EXPLANATION.—We write down without multiplication the square of each term with twice the product of each term multiplied by each of the terms which follow it.

Find the square of

2. $a + b + c$.
3. $x - y + z$.
4. $a + c + d + e$.
5. $1 - a + a^2 - a^3$.

Find the square of

6. $a + 2b + 3c + d$.
7. $1 + 2a - 3a^2 + a^3$.
8. $1 - 2x - y^2 + xy$.
9. $2a - b + c - d$.

CASE IV.

216. Involution of binomials by the Binomial Theorem.

$(a+b)^2 = a^2 + 2ab + b^2.$
$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$
$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$
$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$
$(a-b)^2 = a^2 - 2ab + b^2.$
$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$
$(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4.$
$(a-b)^5 = a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5.$

Examine carefully the above powers of $(a+b)$ and $(a-b)$.

1. How many terms are there in the second power of $(a+b)$? Of $(a-b)$? How many terms in the third power? How many in the fourth power? How many in the fifth power?

2. How does the number of terms in any power compare with the exponent of the power?

3. In what terms is a found in the second power of $(a+b)$? Of $(a-b)$? In the third power? In the fourth power? In the fifth power?

4. In what terms of any power of a binomial is the *first* letter of a binomial found?

5. In what terms of any power of a binomial is the *second* letter of a binomial found?

6. What is the exponent of the first letter in each term of the second power of $(a+b)$? Of $(a-b)$? What in the third power? What in the fourth power? What in the fifth power?

7. What will be the exponent of the first letter of the

binomial in the first term of any power of a binomial? In the second? In the third, etc.?

8. What will be the exponent of the second letter in the second term of any power? In the third term? In the fourth term, etc.?

9. What is the coëfficient of the first and the last terms in any power?

10. How does the coëfficient of the second term compare with the exponent of the power to which the binomial is to be raised?

11. If the coëfficient of the second term is multiplied by the exponent of the first quantity in that term, and divided by the number of the term, or by the exponent of the second quantity in that term increased by 1, what is the result? Of what term is it the coefficient?

12. If the same thing is done to the coëfficient of the third term, what coëfficient is obtained?

13. What are the signs of the terms in all the powers of $(a+b)$?

14. What terms have the *plus* sign in the second power of $(a-b)$? In the third power? In the fourth power? In the fifth power?

15. What terms have the *minus* sign in the second power of $(a-b)$? In the third power? In the fourth power? In the fifth power?

16. What terms are positive and what negative in any power of $(a-b)$?

217. Principles.—1. *The number of terms in the power of any binomial is one more than the exponent of the required power.*

2. *The letter of the first term of the binomial is found in all the terms except the last; the second letter in all the terms*

except the first, and both letters are found in all the terms except the first and last.

3. *The exponent of the letter of the first term of the binomial in the first term of the power is the same as the index of the required power, and decreases by 1 in each term at the right. The exponent of the letter of the second term of the binomial is 1 in the second term, and increases by 1 in each term at the right.*

4. *The coëfficient of the first term is 1. The coëfficient of the second term is the same as the index of the required power. The coëfficient of any term, multiplied by the exponent of the first letter in that term, and divided by the number of the term, or by the exponent of the second letter increased by 1, will be the coëfficient of the next term.*

5. *If both terms of the binomial are positive, all the terms of any power will be positive.*

6. *If the second term of the binomial has the minus sign, all the odd terms counting from the left will be positive, and all the even terms will be negative.*

The sum of the exponents in any term is always equal to the exponent of the required power.

EXAMPLES.

1. Find the fifth power of $(x - y)$ by the binomial theorem.

SOLUTION.

Letters,	x	xy	xy	xy	xy	y
Letters and Exponents,	x^5	x^4y	x^3y^2	x^2y^3	xy^4	y^5
Coëfficients,	1	5	10	10	5	1
Signs,		$-$	$+$	$-$	$+$	$-$

Combined, $x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$

2. Expand $(x+y)^3$.
3. Expand $(a+b)^4$.
4. Expand $(a-c)^5$.
5. Expand $(a-x)^4$.
6. Expand $(a+x)^6$.
7. Expand $(a-c)^9$.
8. Expand $(x-y)^7$.
9. Expand $(x+y)^{10}$.
10. Expand $(x+1)^5$.
11. Expand $(x-1)^6$.
12. Expand $(1+a)^5$.
13. Expand $(1-a)^7$.
14. Expand $(x+ac)^4$.
15. Expand $(x+bc)^5$.

When the terms of the binomial have coëfficients, the binomial may be expanded as follows:

16. Find the third power of $2a^2+3b$.

SOLUTION.

Let $2a^2=x$ and $3b=y$. Then $x+y=2a^2+3b$.

$$(x+y)^3=x^3+3x^2y+3xy^2+y^3.$$

Restoring the values of x and y, we have

$$\begin{aligned}
x^3 &= && 8a^6\\
3x^2y &= 3\times 4a^4\times 3b &&= 36a^4b\\
3xy^2 &= 3\times 2a^2\times 9b^2 &&= 54a^2b^2\\
y^3 &= && 27b^3\\
\hline
(2a^2+3b)^3 &= 8a^6+36a^4b+54a^2b^2+27b^3
\end{aligned}$$

17. What is the fourth power of $2x-3y$?

18. What is the third power of $3a+2c$?

19. What is the fourth power of $\frac{a}{2}+\frac{b}{3}$?

20. What is the fourth power of $2+\frac{x}{3}$?

21. What is the third power of $x-\frac{a}{y}$?

22. What is the fourth power of $2ax+3by$?

23. What is the third power of $2a^2x+3b^2y^2$?

EVOLUTION.

218. 1. What are the factors of 16? What are the two equal factors of 16? Of 36? Of 49? Of 81?

2. What is one of the two equal factors of a^2, or what is the second root of a^2? $4a^2$? $16a^2$? $25a^4$? $36a^4$? $81a^2$? $121x^2$?

3. What quantity used three times as a factor will produce a^3, or what quantity is the third root of a^3? Of $8a^3$? Of $27a^3$? Of $8a^6$? Of $64a^6$?

4. What is the product of -2 multiplied by -2? What of $+2$ multiplied by $+2$? Of $-2 \times -2 \times -2$? Of $+2 \times +2 \times +2$?

5. When any number of positive factors is multiplied together, what is the sign of the product?

6. When an *even* number of negative factors is multiplied together, what is the sign of the product? What is the sign of the product when an *odd* number of factors is multiplied together?

7. Since a power which has a negative sign is the product of an odd number of equal factors each having the *minus* sign, what is the sign of an *odd* root of any *negative* quantity?

8. Since a power having the plus sign may oe the product of an even or of an odd number of positive equal factors, and can not be the product of an odd number of negative equal factors, what is the sign of an *odd* root of a positive quantity?

9. Since a power having the plus sign may be the product of an even number of either positive or negative factors, what is the sign of an *even* root of a positive quantity?

10. What quantity used twice as a factor will give a product with the negative sign? What used four times as a factor? What used six times as a factor?

11. What may be said, then, regarding an *even* root of a negative quantity?

DEFINITIONS.

219. A **Root** of a quantity is one of the equal factors of the quantity.

Thus, $4a$ is a root of $16a^2$, and $3a$ of $27a^3$.

Roots are named as follows:

One of two equal factors is called the *second*, or *square root*.
One of three equal factors is called the *third*, or *cube root*.
One of four equal factors is called the *fourth root*.
One of five equal factors is called the *fifth root*, etc.

220. **Evolution** is the process of finding a root of a quantity. It is also called the process of *extracting a root* of a quantity.

221. The **Radical, or Root, Sign** is $\sqrt{}$. When it is placed before a quantity, it shows that a root of the quantity is required.

222. The **Index** of the root is a figure, or quantity, written in the opening of the radical sign to show what root is sought.

When no index is written the *square* root is indicated.

Thus, $\sqrt{a+b}$ shows that the square root of $(a+b)$ is sought, and $\sqrt[5]{a^2+y}$ indicates that the fifth root of (a^2+y) is to be found.

223. Fractional exponents also are used to indicate roots.

Thus, $a^{\frac{1}{3}}$ means the third root of a; $a^{\frac{1}{8}}$ means the eighth root of a.

224. In fractional exponents the *numerator* indicates the *power*, and the *denominator* the root. For, since in raising a quantity to a given power we multiply the exponent of the quantity by the exponent of the power to which it is to be raised, in finding a root the exponent of the quantity should be divided by the index of the root.

Thus, $a^{\frac{2}{3}}$ means the third root of a^2; $a^{\frac{5}{7}}$ the seventh root of a^5.

225. PRINCIPLES.—1. *An odd root of a quantity has the same sign as the quantity itself.*

2. *An even root of a positive quantity is either positive or negative.*

3. *An even root of a negative quantity is impossible or imaginary.*

CASE I.

226. Evolution of perfect powers by factoring.

1. What is the square root of $16a^4b^2c^6$?

PROCESS.

$$16a^4b^2c^6 = 2\times 2\times 2\times 2,\ a,\ a,\ a,\ a,\ b,\ b,\ c,\ c,\ c,\ c,\ c,\ c.$$
$$2\times 2,\ a,\ a,\ b,\ c,\ c,\ c = 4a^2bc^3.$$

EXPLANATION.—Since the square root of a quantity is one of its two equal factors, the square root of $16a^4b^2c^6$ may be found by separating it into its prime factors, and finding the product of the quantities forming one of the two equal sets of factors. Separating into prime factors, one of the two equal sets of factors is found to be 2×2, a, a, b, c, c, c, which is equal to $4a^2bc^3$.

RULE.—*Separate the quantities into their prime factors. Arrange these into as many sets containing the same factors as there are units in the degree of the required root. The product of the factors which form a set will be the root.*

2. Find the square root of $16x^2y^2z^2$. Of $36a^2b^4c^2$.
3. Find the cube root of $8x^3y^6z^6$. Of $27b^3z^6y^3$.
4. Find the cube root of $64x^6y^3z^3$. Of $27a^6x^6z^3$.
5. Find the square root of 144. Of 256. Of 324.
6. Find the cube root of 64. Of 512. Of 4096.
7. Find the fourth root of 1296. The fifth root of 248832.
8. Find the square root of $a^2+2ab+b^2$. Of a^2+2a+1.
9. Find the square root of $a^4+2a^3b+a^2b^2$. Of $m^2n^2+2m^3n+m^4$.

CASE II.

227. Evolution of any monomial.

1. What is the square root of $25x^8y^6$?

PROCESS.

$$\sqrt{25x^8y^6} = \pm 5x^4y^3$$

EXPLANATION.—Since, to square a monomial, we square the coëfficient and multiply the exponents of the letters by 2, to extract the square root we must extract the square root of the coëfficient and divide the exponents of the letters by 2. Since the even root of a positive quantity is either positive or negative (Prin. 2), the sign of the root is either plus or minus. Hence, the square root of the quantity is $\pm 5x^4y^3$.

RULE.—*Extract the required root of the numerical coëfficient; divide the exponent of each letter by the index of the root sought, and prefix the proper sign to the result.*

The root of a fraction is found by taking the root of the numerator and of the denominator.

2. What is the square root of $16a^2b^4c^4$?

3. What is the cube root of $-8a^3b^6c^3$?

4. What is the square root of $4a^4c^4x^2$?

5. What is the cube root of $27x^3y^6z^3$?

6. What is the fourth root of $16a^4b^8c^8$?

7. What is the cube root of $-8ab^3c^2$?

8. What is the fifth root of $-a^5c^{10}x^2y$?

9. What is the mth root of $a^mx^{3m}y^m$?

10. Find the required root of $\sqrt[3]{a^3x^8y^2z^5}$?

11. Find the required root of $\sqrt[4]{x^4y^8z^8w^2}$?

12. Find the required root of $\sqrt[4]{16x^{4n}y^{2n}z}$?

13. Find the required root of $(a^4x^2y^8z^4)^{\frac{1}{2}}$?

14. Find the required root of $(a^{2n}x^{4n}y^{3n})^{\frac{1}{n}}$?

15. What is the square root of $\dfrac{16a^2}{25y^4}$?

16. What is the cube root of $\dfrac{8x^3}{27y^6}$?

17. What is the cube root of $\dfrac{125x^3y^3}{216a^6y^3}$?

CASE III.

228. To extract the square root of a polynomial.

1. What is the square of $(a+b)$?

2. Since $a^2+2ab+b^2$ is the square of $(a+b)$, what is the square root of $a^2+2ab+b^2$?

3. How may the first term of the square root be found from $a^2+2ab+b^2$?

4. How may the second term of the square root be found from the second term of the power, $2ab$?

5. What are the factors of $2ab + b^2$?

6. Since $2ab + b^2$ is equal to $b(2a + b)$, what are the factors of the last two terms of the square of a binomial?

EXAMPLES.

1. Find the square root of $a^2 + 2ab + b^2$.

PROCESS.

$$\begin{array}{lll} & a^2 + 2ab + b^2 & \underline{|\ a + b\ } \\ & \underline{a^2} & \\ \text{Trial divisor, } 2a & \quad |\ 2ab + b^2 & \\ \text{Complete divisor, } 2a + b & \quad |\ \underline{2ab + b^2} & \end{array}$$

EXPLANATION.—Since, if the quantity is a square, one of its terms is a perfect square, the first term of the root is the square root of a^2, which is a.

Subtracting a^2 from the entire quantity, there is left $2ab + b^2$.

Since the second term of the root can be obtained from the first term of the remainder by dividing it by twice the term of the root already found, the second term of the root of this quantity will be found by dividing $2ab$ by $2a$, the trial divisor, which gives b for the second term of the root. And since the last two terms of the square of a binomial consist of the product of the second term of the root multiplied by twice the first plus or minus the second, $2a + b$ is the entire quantity or *complete divisor*, which is to be multiplied by b. Multiplying by b and subtracting, there is no remainder.

Therefore, $a + b$ is the square root of the quantity.

Since, in squaring $a + b + c$, $a + b$ may be represented by x, the square will be $x^2 + 2xc + c^2$. Hence, it is obvious that the square root of a quantity, whose root consists of *more than two terms*, may be extracted in the same way as in example 1, by considering the terms already found as one term.

2. Find the square root of $x^4 + 4x^3 - 6x^2 - 20x + 25$.

PROCESS.

$$
\begin{array}{l|l|l}
 & x^4 + 4x^3 - 6x^2 - 20x + 25 & \underline{x^2 + 2x - 5} \\
 & x^4 & \\
\cline{2-2}
\underline{2x^2 + 2x} & \quad 4x^3 - 6x^2 & \\
 & \quad 4x^3 + 4x^2 & \\
\cline{2-2}
\underline{2x^2 + 4x - 5} & \quad -10x^2 - 20x + 25 & \\
 & \quad \underline{-10x^2 - 20x + 25} &
\end{array}
$$

EXPLANATION.—By proceeding as in the previous example, the first two terms of the root are found to be $x^2 + 2x$.

To find the next term of the root, we consider $x^2 + 2x$ as one quantity, which we multiply by 2 for the trial divisor. Dividing the first term of the remainder by the first term of the divisor, the third term of the root is obtained, which is -5. Annexing this, as before, to the trial divisor already found, the entire divisor is $2x^2 + 4x - 5$. This multiplied by -5, and subtracted from $-10x^2 - 20x + 25$, leaves no remainder.

Hence, the square root of the quantity, $x^4 + 4x^3 - 6x^2 - 20x + 25$ is $x^2 + 2x - 5$.

RULE.—*Arrange the terms of the polynomial with reference to the powers of some letter.*

Extract the square root of the first term, write the result as the first term of the root, and subtract its square from the given polynomial.

Divide the first term of the remainder by twice the root already found, as a trial divisor, and the quotient will be the next term of the root. Write this result in the root, and annex it to the trial divisor, to form a complete divisor.

Multiply the complete divisor by this term of the root, and subtract the product from the first remainder.

Continue in this manner until all the terms of the root are found.

Find the square root

3. Of $x^2 + 4x + 4$.
4. Of $b^2 + 2bx + x^2$.
5. Of $4x^2 + 4x + 1$.
6. Of $a^2 + ab + \frac{1}{4}b^2$.
7. Of $9a^2 - 12ab + 4b^2$.
8. Of $a^2 + 2ab + b^2 - 2ac - 2bc + c^2$.
9. Of $4x^4 - 12x^3 + 13x^2 - 6x + 1$.
10. Of $4a^4 + 4a^3 - 7a^2 - 4a + 4$.
11. Of $x^6 - 4x^5 + 10x^4 - 12x^3 + 9x^2$.
12. Of $16a^4 - 24a^3x + 49a^2x^2 - 30ax^3 + 25x^4$.
13. Of $a^2 + b^2 + c^2 - 2ab - 2ac + 2bc$.
14. Of $40x^2 - 12x^3 + 9x^4 - 24x + 36$.
15. Of $4x^6 + 5x^4 + 12x^5 - 5x^2 - 10x^3 + 2x + 1$.
16. Of $49x^4 - 28x^3 - 17x^2 + 6x + \dfrac{9}{4}$.

SQUARE ROOT OF NUMBERS.

$1^2 = 1$	$10^2 = 100$	$100^2 = 10000$
$9^2 = 81$	$99^2 = 9801$	$999^2 = 998001$

229. 1. How many figures are required to express the square of any number of *units?*

2. How does the number of figures required to express the second power of any number of *tens* compare with the number of figures expressing the tens? How does the number of figures expressing the second power of *hundreds* compare with the number of figures expressing hundreds?

3. If the second power of a number is expressed by 3 figures, how many orders of units are there in the number? If by 4, how many? If by 5, how many? If by 7, how many?

4. How, therefore, may the number of figures in the square root of any number be found?

230. PRINCIPLES.—1. *The square of a number is expressed by twice as many figures as is the number itself, or one less.*

2. *The orders of units in the square root of a number correspond to the number of periods of two figures each into which the number can be separated, beginning at units.*

231. Any number may be separated into its parts and squared, according to the principles already given. Thus, if any number consisting of tens and units is separated into tens and units, and raised to the second power, the result will be the *tens* $^2 + 2$ times the *tens* $\times$ the *units* $+$ the *units* ^{2}. Or, representing the tens by t and the units by u, the result may be expressed as follows:

PRINCIPLE.—*The square of a number consisting of tens and units is equal to* $t^2 + 2tu + u^2$.

Thus, $35 = 3$ tens plus 5, or $30 + 5$ and $35^2 = 30^2 + 2\ (30 \times 5) + 5^2 = 1225$.

EXAMPLES.

1. What is the square root of 1225?

FIRST PROCESS.

$$
\begin{array}{ll|l}
 & 12{\cdot}25 & 30+5 \\
t^2 = & 900 & \\
\cline{2-2}
 & 325 & \\
2t \quad = 60 & & \\
u \quad\; = 5 & & \\
\hline
2t + u = 65 & 325 & \\
\end{array}
$$

EXPLANATION. — According to Prin. 2, Art. 230, the orders of units in the square root of a number may be determined by separating the number into periods of two figures each, beginning at units. Separating 1225 thus, there are found to be two orders of units in the root—that is, it is composed of tens and units. Since the square of tens is hundreds, 12 hundreds must be the square of at least 3 tens. 3 tens, or 30 squared, are 900, and 900 subtracted from 1225 leaves 325, which is equal to 2 times the tens $\times$ the units $+$ the units ^{2},

Since two times the tens multiplied by the units is much greater than the square of the units, 325 is *nearly* 2 times the tens multiplied by the units. Therefore, if 325 is divided by 2 times the tens, or 60, the quotient will be approximately the units of the root. Dividing by 60, the *trial divisor*, the units are found to be 5. And since, the complete divisor is found by adding the units to twice the tens, the complete divisor is $60 + 5$, or 65. This multiplied by 5 gives as a product 325. Therefore, the square root of 1225 is 35.

SECOND PROCESS.

		12·25	35
$t^2 =$		9	
$2t$	$= 60$	325	
u	$= 5$		
$2t + u$	$= 65$	325	

EXPLANATION. — In practice it is usual to place the figures of the same order in a column, and to disregard the ciphers on the right of the products.

Since any number may be regarded as composed of tens and units, the processes given above have a general application.

Thus, $325 = 32$ tens $+ 5$ units; $4685 = 468$ tens $+ 5$ units.

2. Find the square root of 137641.

PROCESS.

	13·76·41	371
	9	
Trial divisor $= 2 \times 30 = 60$	476	
Complete divisor $= 60 + 7 = 67$	469	
Trial divisor $= 2 \times 370 = 740$	741	
Complete divisor $= 740 + 1 = 741$	741	

RULE.—*Separate the number into periods of two figures each, beginning at units.*

Find the greatest square in the left-hand period, and write its root for the first figure of the required root.

Square this root and subtract the result from the left-hand period, and annex to the remainder the next period for a new dividend.

Double the root already found, with a cipher annexed, for a trial divisor, and by it divide the dividend. The quotient, or quotient diminished, will be the second figure of the root. Add to the trial divisor the figure last found, multiply this complete divisor by the figure of the root found, subtract the product from the dividend, and to the remainder annex the next period for the next dividend.

Proceed in this manner until all the periods have been used thus. The result will be the square root sought.

1. When the number is not a perfect square annex periods of ciphers and continue the process.

2. Decimals are pointed off into periods of two figures each, by beginning at tenths and passing to the right.

3. The square root of a common fraction may be found by extracting the square root of both numerator and denominator separately, or by reducing it to a decimal and then extracting its root.

(For explanation with blocks, see Practical Arithmetic, page 322.)

Extract the square root of the following:

3. 2809.	7. 70756.	11. 938961.
4. 3969.	8. 118336.	12. 5875776.
5. 4356.	9. 674041.	13. 12574116.
6. 9216.	10. 784996.	14. 30858025.

15. Find the value of $\sqrt{222784}$.
16. Find the value of $\sqrt{11390625}$.
17. Find the value of $\sqrt{.763876}$.
18. Find the value of $\sqrt{.30858025}$.
19. What is the square root of .093636?
20. What is the square root of .099225?
21. What is the square root of .7075613?

22. What is the square root of $\frac{55225}{784996}$?

23. What is the square root of $\frac{781456}{978121}$?

24. What is the square root of $\frac{1}{2}$ to five decimal places?

25. What is the square root of $\frac{3}{4}$ to five decimal places?

26. What is the square root of .9 to five decimal places?

CASE IV.

232. To extract the cube root of a polynomial.

1. What is the cube of $(a+b)$?

2. Since $a^3+3a^2b+3ab^2+b^3$ is the cube of $(a+b)$, what is the cube root of $a^3+3a^2b+3ab^2+b^3$?

3. How may the first term of the root be found from $a^3+3a^2b+3ab^2+b^3$?

4. How may the second term of the root be found from the second term of the power, $3a^2b$?

5. What are the factors of $3a^2b+3ab^2+b^3$?

6. Since $3a^2b+3ab^2+b^3$ is equal to $b(3a^2+3ab+b^2)$, what are the factors of the last three terms of the cube of a binomial?

EXAMPLES.

1. Find the cube root of $a^3+3a^2b+3ab^2+b^3$.

PROCESS.

	$a^3+3a^2b+3ab^2+b^3$	$a+b$
	a^3	
Trial divisor, $3a^2$	$3a^2b+3ab^2+b^3$	
Complete divisor, $3a^2+3ab+b^2$	$3a^2b+3ab^2+b^3$	

EXPLANATION.—Since, if the quantity is a cube one of the terms is a cube, the first term of the root is the cube root of a^3, which is a. Subtracting a^3 from the entire quantity, there is left $3a^2b+3ab^2+b^3$.

Since the second term of the root can be obtained from the first term of the remainder, by dividing it by three times the square of the root already found, the second term of the root of the quantity will be found by dividing $3a^2b$ by $3a^2$, the *trial divisor*, which gives b for the second term of the root. And since the last three terms of the cube of a binomial consist of the product of the second term of the root with 3 times the square of the first, 3 times the product of the first and second, and the square of the second, $3a^2+3ab+b^2$, is the entire quantity, or *complete divisor*, which is to be multiplied by b.

Multiplying by b, and subtracting, there is no remainder. Therefore, $a+b$ is the cube root of $a^3+3a^2b+3ab^2+b^3$.

Since, in cubing $a+b+c$, $a+b$ may be expressed by x, the cube will be $x^3+3x^2c+3xc^2+c^3$. Hence, it is obvious that the cube root of a quantity, whose root consists of *more than two terms*, may be extracted in the same way as in example 1, by considering the terms already found as one term.

2. Find the cube root of $x^6-3x^5+5x^3-3x-1$.

PROCESS.

$$
\begin{array}{lr|l}
 & x^6-3x^5+5x^3-3x-1 & \underline{x^2-x-1} \\
 & \underline{x^6\qquad\qquad\qquad\qquad} & \\
\text{Trial divisor, } 3x^4 & -3x^5+5x^3 & \\
\text{Complete divisor, } \underline{3x^4-3x^3+x^2} & \underline{-3x^5+3x^4-x^3} & \\
\text{Trial divisor, } 3x^4-6x^3+3x^2 & -3x^4+6x^3-3x-1 & \\
\text{Complete divisor, } 3x^4-6x^3+3x+1 & \underline{-3x^4+6x^3-3x-1} &
\end{array}
$$

EXPLANATION.—The first two terms are found in the same manner as in the previous example. To find the next term x^2-x is considered as one quantity, which we square and multiply by 3 for a trial divisor. Dividing the remainder by this trial divisor, the next term of the root is found to be -1. Adding to this trial divisor 3 times (x^2-x), multiplied by -1 and the square of -1, we obtain the complete divisor. This, multiplied by -1, and subtracted, leaves no remainder. Hence, the cube root of the quantity is x^2-x-1.

RULE.—*Arrange the polynomial with reference to the powers of some letter.*

Extract the cube root of the first term, write the result as the first term of the quotient, and subtract its cube from the given polynomial.

Divide the first term of the remainder by 3 times the square of the root already found, as a trial divisor, and the quotient will be the next term of the root.

Add to this trial divisor 3 times the product of the first and second terms of the root, and the square of the second term. The result will be the complete divisor.

Multiply the complete divisor by the last term of the root found, and subtract this product from the dividend. Continue in this manner until all the terms of the root are found.

Each succeeding *term of the root* may be obtained by dividing the first term of each successive remainder by 3 times the square of the first term of the root.

Find the cube root

3. Of $x^3+6x^2y+12xy^2+8y^3$.
4. Of $27a^3+27a^2+9a+1$.
5. Of $8x^3-36x^2+54x-27$.
6. Of $27x^3+108x^2+144x+64$.
7. Of $a^3+3a+\frac{3}{a}+\frac{1}{a^3}$.
8. Of $a^3-12a^2+48a-64$.
9. Of $8a^3+12a^2+6a+1$.
10. Of $27x^6-54x^5+63x^4-44x^3+21x^2-6x+1$.
11. Of $8m^6-36m^5+66m^4-63m^3+33m^2-9m+1$.
12. Of $1-3a+6a^2-7a^3+6a^4-3a^5+a^6$.
13. Of $m^3-3m^2+5-\frac{3}{m^2}-\frac{1}{m^3}$.
14. Of $x^3-3x^2y-y^3+8z^3+6x^2z-12xyz+6y^2z+12xz^2-12yz^2+3xy^2$.

Any root of a polynomial may be extracted in the same manner as the cube and square roots have been extracted. Thus, to find a formula for obtaining the complete divisor in extracting the *fourth*, *fifth*, *sixth*, or *any* required root of a polynomial, raise $(a+b)$ to the required power, and separate all the terms after the first into two factors, one of which shall be the first power of the second term of the root. The other factor will be the formula for the complete divisor.

$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$
Trial divisor, $5a^4$.
Complete divisor, $(5a^4 + 10a^3b + 10a^2b^2 + 5ab^3 + b^4)$.

$(a+b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7.$
Trial divisor, $7a^6$.
Complete divisor, $(7a^6 + 21a^5b + 35a^4b^2 + 35a^3b^3 + 21a^2b^4 + 7ab^5 + b^6)$.

Since the *fourth* power is the *square* of the *second* power, the *sixth* power the *cube* of the *second* power, etc., any root whose index is composed of the factors 2 or 3, or 2 and 3, may be found by extracting successively the roots corresponding to the factors of the index.

Thus, the fourth root may be obtained by extracting the square root of the square root; the sixth root, by extracting the cube root of the square root; the eighth root, by extracting the square root of the square root of the square root; the ninth root, by extracting the cube root of the cube root.

16. Find the fourth root of $1 - 8a + 24a^2 - 32a^3 + 16a^4$.
17. Find the fifth root of $x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$.
18. Find the sixth root of $x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1$.

CUBE ROOT OF NUMBERS.

$1^3 = 1$	$10^3 = 1000$	$100^3 = 1000000$
$9^3 = 729$	$99^3 = 970299$	$999^3 = 997002999$

233. 1. How many figures are required to express the cube of any number of *units?*

2. How does the number of figures required to express the cube of any number of *tens* compare with the number of figures expressing the tens? How does the number of figures expressing the cube of any number of hundreds compare with the number of figures expressing hundreds?

3. If, then, the cube of a number is expressed by 4 figures, how many orders of units are there in the root? If by 5 figures, how many? If by 6 figures, how many? If by 8 figures, how many?

4. How may the number of figures in the cube root of a number be found?

234. PRINCIPLES.—1. *The cube of a number is expressed by three times as many figures as the number itself, or one or two less.*

2. *The orders of units in the cube root of a number correspond to the number of periods of three figures each into which the number can be separated, beginning at units.*

235. Any number may be separated into its parts, and cubed according to the principles already given. Hence, if any number consisting of tens and units is separated into tens and units, and raised to the third power, the result will be the $tens^3$ + 3 times the $tens^2$ × the *units* + 3 times the *tens* × $units^2$ + the $units^3$.

Or, representing the tens by t and the units by u, the result may be expressed as follows:

PRINCIPLE.—*The cube of a number consisting of tens and units is equal to* $t^3 + 3t^2u + 3tu^2 + u^3$.

Thus, $35 = 3$ tens $+ 5$ units, or $30 + 5$, and $35^3 = 30^3 + 3(30^2 \times 5) + 3(30 + 5^2) + 5^3 = 42875$.

EXAMPLES.

1. What is the cube root of 13824?

FIRST PROCESS.

$$
\begin{array}{lrr|l}
 & & 13{\cdot}824 &)\,\underline{20 + 4} \\
 & t^3 = & 8000 & \\
\text{Trial divisor,} & 3t^2 = & 1200 & 5824 \\
 & 3tu = & 240 & \\
 & u^2 = & 16 & \\
\text{Complete divisor,} & & = 1456 & 5824
\end{array}
$$

EXPLANATION.—According to Prin. 2, Art. 234, the orders of units may be determined by separating the number into periods of three figures each, beginning at units. Separating 13824 thus, there are found to be two orders of units in the root; that is, it is composed of tens and units.

Since the cube of tens is thousands, 13 thousands must be the cube of at least 2 tens. 2 tens, or 20 cubed, is 8000, and 8000 subtracted from 13824 leaves 5824, which is equal to three times the tens^2 $\times$ the units $+$ 3 times the tens $\times$ the units^2 $+$ the units^3.

Since three times the tens^2 is much greater than 3 times the tens $\times$ the units^2, 5824 is a little more than three times the tens^2 $\times$ the units. If, then, 5824 is divided by 3 times the tens^2, or 1200, the trial divisor, the quotient 4 will be the units of the root, provided proper allowance has been made for the additions to be made to obtain the complete divisor.

Since the complete divisor is found by adding to 3 times the tens^2, 3 times the tens $\times$ the units and the units^2, the complete divisor will be $1200 + 240 + 16$, or 1456. This multiplied by 4 gives as a product, 5824. Therefore, the cube root of 13824 is 24.

SECOND PROCESS.

$$
\begin{array}{rr|l}
 & 13\cdot824 & \underline{24} \\
t^3 = & 8 & \\
\hline
3t^2 = 1200 & 5824 & \\
3tu = 240 & & \\
u^2 = 16 & & \\
\hline
1456 & 5824 &
\end{array}
$$

EXPLANATION.—In practice it is usual to place figures of the same order in a column, and to disregard the ciphers on the right of the products.

Since any number may be regarded as composed of tens and units, the processes given have a general application.

Thus, $468 = 46$ tens $+ 8$ units; $3829 = 382$ tens $+ 9$ units.

2. What is the cube root of 48228544?

PROCESS.

$$
\begin{array}{lrr|l}
 & & 48\cdot228\cdot544 & \underline{364} \\
 & & 27 & \\
\text{Trial divisor} = 3(30)^2 & = 2700 & 21228 & \\
3(30 \times 6) & = 540 & & \\
6^2 & = 36 & & \\
\text{Complete divisor} & = 3276 & 19656 & \\
\hline
\text{Trial divisor} = 3(360)^2 & = 388800 & 1572544 & \\
3(360 \times 4) & = 4320 & & \\
4^2 & = 16 & & \\
\text{Complete divisor} & = 393136 & 1572544 &
\end{array}
$$

RULE.—*Separate the number into periods of three figures each, beginning at units.*

Find the greatest cube in the left-hand period, and write its root for the first figure of the required root. Cube the root, subtract the result from the left-hand period, and annex to the remainder the next period for a dividend.

Take 3 times the square of the root already found with a

cipher annexed, for a trial divisor, and by it divide the dividend. The quotient, or quotient diminished, will be the second figure of the root.

To this trial divisor add 3 times the product of the first part of the root with a cipher annexed, by the second part, and also the square of the second part. Their sum will be the complete divisor.

Multiply the complete divisor by the second part of the root, and subtract the product from the dividend. Continue thus until all the figures of the root have been found.

1. When there is a remainder after subtracting the last product, annex decimal ciphers and continue the process.

2. Decimals are pointed off into periods of three figures each, by beginning at tenths and passing to the right.

3. The cube root of a common fraction may be found by extracting the cube root of both numerator and denominator separately, or by reducing it to a decimal and then extracting its root.

4. A rule for the extraction of any root may be formed from the general formulas, as is shown on page 190.

5. For explanation with blocks, see Practical Arithmetic, page 330.

Extract the cube root of the following:

3. 74088.	6. 704969.	9. 5545233.
4. 262144.	7. 185193.	10. 2000376.
5. 166375.	8. 250047.	11. 153990656.

12. Find the cube root of 2 to 3 decimal places.
13. Find the cube root of 9 to 4 decimal places.
14. Find the cube root of .27. Of .64.
15. Find the cube root of $\frac{2}{3}$.
16. Find the cube root of $\frac{3}{5}$.
17. Find the fourth root of 145161.
18. Find the fifth root of 248832.

RADICAL QUANTITIES.

236. 1. In what two ways may the root of a quantity be indicated?

2. What is indicated by $\sqrt{x^2}$? By $(3a)^{\frac{1}{3}}$? By $5b^{\frac{1}{4}}$?

3. In the expression $5\sqrt{x^3}$, what shows how many times $\sqrt{x^3}$ is taken?

4. What is that called which shows how many times a quantity is taken?

5. To what quantity without the radical sign is $\sqrt{9x^2}$ exactly equal? $\sqrt{36b^2}$? $\sqrt{49m^2}$? $\sqrt[3]{27a^3}$?

6. To what quantity without the radical sign is $\sqrt{7a}$ exactly equal? $\sqrt{8x}$? $\sqrt{12x^3}$? $\sqrt[3]{15a^2}$? $\sqrt[3]{10x^2}$?

7. What is the square root of 9? Of 4? Of 9×4?
What is the square root of a^2? Of b^2? Of $a^2 \times b^2$?
What is the cube root of 8? Of 27? Of 8×27?

8. How does the product of the roots of two quantities compare with the root of the product of the quantities?

9. How, then, does the root of a quantity compare with the product of the roots of the factors?

DEFINITIONS.

237. A **Radical Quantity**, or **Radical**, is a quantity whose root is indicated.

The root may be indicated by the *radical sign* or by a *fractional exponent.*

Thus, $\sqrt{7x^2}$, $\sqrt{25a^2}$, $\sqrt[3]{4x}$, $x^{\frac{1}{3}}$, $5x^{\frac{1}{4}}$, and $4a^{\frac{2}{3}}$ are radical quantities.

238. The **Coëfficient** of a radical is the quantity prefixed to the radical part to show how many times it is taken.

Thus, 5, a, and bc are, severally, the coëfficients of the expressions $5\sqrt{x}$, $a\sqrt{x^2y}$, $bcx^{\frac{1}{4}}$.

239. The **Degree** of a radical is indicated by the index of the radical or the denominator of the fractional exponent.

Thus, $\sqrt{x}$, $ax^{\frac{1}{2}}$, $(x+y)^{\frac{1}{2}}$, are radicals of the *second* degree.
$\sqrt[3]{y^2}$, $\sqrt[3]{a+b}$, $(c+d)^{\frac{1}{3}}$, are radicals of the *third* degree.

240. Similar Radicals are those which have the same quantity under the radical sign, with the same index, or which have the same fractional exponent.

Thus, $5\sqrt[3]{x^2y}$, $a\sqrt[3]{x^2y}$, $5b(x^2y)^{\frac{1}{3}}$, are similar radicals.

241. A **Rational Quantity** is a quantity whose indicated root can be extracted exactly.

Thus, $\sqrt{49x^2}$, $\sqrt[3]{27y^3}$, $(a^2+2ab+b^2)^{\frac{1}{2}}$, are rational quantities.

242. A **Surd,** or **Irrational Quantity,** is a quantity whose indicated root can not be extracted exactly.

Thus, $\sqrt{3x}$, $\sqrt[3]{7x^2}$, $(a^2+b^2)^{\frac{1}{2}}$, are irrational quantities.

243. PRINCIPLE.—*The root of any quantity is equal to the product of the roots of its factors.*

REDUCTION OF RADICALS.

CASE I.

244. To reduce a radical to its simplest form.

When a radical contains no factor which is a perfect power corresponding to the degree of the radical, it is in its *simplest form.*

1. Reduce $\sqrt{16a^2x}$ to its simplest form.

PROCESS.

$$\sqrt{16a^2x}=\sqrt{16a^2}\times\sqrt{x}=4a\sqrt{x}$$

EXPLANATION.—Since a radical is in its simplest form when it contains no factor which is a perfect power corresponding to the degree of the radical, we separate $16a^2x$ into two factors, one of which is a perfect square, since the radical is of the second degree. The factors are $16a^2$ and x. Then, by Principle (Art. 243), $\sqrt{16a^2x}=\sqrt{16a^2}\times\sqrt{x}$; or, extracting the root of the perfect square, it becomes $4a\sqrt{x}$.

RULE.—*Separate the quantity under the radical sign into two factors, one of which shall contain all the perfect powers of the radical corresponding in degree with the radical. Extract the root of the rational factor, multiply it by the coëfficient of the given radical, and place the product as the coëfficient of the surd.*

Reduce the following radicals to their simplest form:

2. $\sqrt{18x^2}$.	7. $\sqrt{81a^2xy}$.
3. $\sqrt{36a^2b}$.	8. $3\sqrt{4xy^2z^2}$.
4. $\sqrt{75x^3y^4}$.	9. $5\sqrt{18x^3y^2z^2}$.
5. $\sqrt{100a^6b^3}$.	10. $\sqrt{a^2-a^2x}$.
6. $\sqrt{432a^3x^2y}$.	11. $\sqrt[3]{x^4-x^3y^2}$.

12. $a\sqrt[3]{a^4 - a^3x^2}$.

13. $a\sqrt{a^3 + 2a^2b + ab^2}$.

14. $(a+b)\sqrt{a^4 - a^2b^2}$.

15. $(x+y)\sqrt{x^3 - 2x^2y + xy^2}$.

16. $(x^2 + x^2y + x^2y^2)^{\frac{1}{2}}$.

17. $4a(x + 2xy + xy^2)^{\frac{1}{2}}$.

245. A Fractional Radical Quantity is in its *simplest form*, when only a surd without a denominator is left under the radical.

18. Reduce $\sqrt{\dfrac{3a^2}{5b}}$ to its simplest form.

PROCESS.

$$\sqrt{\frac{3a^2}{5b}} = \sqrt{\frac{3a^2 \times 5b}{5b \times 5b}} =$$

$$\sqrt{\frac{15a^2b}{25b^2}} = \sqrt{15b \times \frac{a^2}{25b^2}} =$$

$$\frac{a}{5b}\sqrt{15b}$$

EXPLANATION. — Since the denominator is to be removed, it must be made a perfect square. This can be done by multiplying the terms of the fraction by $5b$. Then factoring, as in previous examples, and extracting the root of the rational part, the result is $\dfrac{a}{5b}\sqrt{15b}$, in which the radical is in its simplest form.

RULE.—*Multiply the terms of the fraction by a quantity which will make its denominator a perfect power of the same degree as the given root.*

Simplify this result according to the previous rule.

Reduce the following to their simplest form:

19. $\sqrt{\dfrac{2}{3}}$.

20. $\sqrt{\dfrac{3}{7}}$.

21. $\sqrt[3]{\dfrac{5}{9}}$.

22. $\sqrt{\dfrac{5a}{7b}}$.

23. $\sqrt{\dfrac{5xy}{6ab}}$.

24. $\sqrt[3]{\dfrac{7a}{8b}}$.

25. $\sqrt{\dfrac{3x^3}{5a}}$.

26. $2\sqrt{\dfrac{5}{8}}$.

27. $2\sqrt{\frac{3x^4}{4y^2}}$.

28. $(a+b)\sqrt{\frac{a}{a-b}}$.

29. $(x+y)\sqrt[3]{\frac{x}{(x+y)^2}}$.

30. $(a^2-b^2)\sqrt{\frac{x}{a^2+b^2}}$.

31. $3a\left(\frac{3x}{7y}\right)^{\frac{1}{2}}$.

32. $5xy\left(\frac{3x^2y}{4x^2y^2}\right)^{\frac{1}{3}}$.

CASE II.

246. To reduce a rational quantity to the form of a radical.

1. What is the root of $\sqrt{4a^2}$? To what quantity under the radical sign of the second degree is $2a$ equal?

2. What is the value of $\sqrt[3]{27a^3}$? To what quantity under the radical sign of the third degree is $3a$ equal?

EXAMPLES.

1. Reduce $6a^2xy$ to the form of the cube root.

PROCESS.

$$6a^2xy = \sqrt[3]{(6a^2xy)^3} = \sqrt[3]{216a^6x^3y^3}$$

EXPLANATION.—Since any quantity is equal to the cube root of its cube, $6a^2xy$ is equal to the cube root of $(6a^2xy)^3$, or $\sqrt[3]{216a^6x^3y^3}$.

RULE.—*Raise the quantity to the power corresponding to the index of the given root, and place the result under the appropriate radical sign.*

The coëfficient of a radical quantity may be placed under the radical sign by raising the coëfficient to the power indicated by the index of the radical, and multiplying the quantity under the radical by the result.

2. Reduce $3ax^2$ to the form of the square root.
3. Reduce $4x^2y$ to the form of the square root.
4. Reduce $2a^2x^2y$ to the form of the cube root.
5. Reduce $a+b$ to the form of the square root.
6. Reduce $a-b$ to the form of the cube root.
7. Express $2a\sqrt{ab}$ entirely under the radical.
8. Express $3b\sqrt{bxy}$ entirely under the radical.
9. Express $2x\sqrt{x^2+y^2}$ entirely under the radical.
10. Express $(x+y)\sqrt{2x}$ entirely under the radical.
11. Express $3a\sqrt{a^2-b^2}$ entirely under the radical.
12. Express $(a-b)\sqrt{a+b}$ entirely under the radical.
13. Express $(x-y)\sqrt{\dfrac{a}{x-y}}$ entirely under the radical.
14. Express $(x^2-y^2)\sqrt{\dfrac{x}{(x-y)^2}}$ entirely under the radical.

CASE III.

247. To reduce radicals to equivalent radicals of the same degree.

1. To what fraction in lower terms is $\frac{3}{6}$ equal? By what equivalent expression having an exponent in lower terms can $a^{\frac{3}{6}}$ be expressed? By what $a^{\frac{2}{6}}$? By what $a^{\frac{2}{8}}$? By what $x^{\frac{3}{9}}$?

2. Express $a^{\frac{1}{2}}$ by an equivalent expression, with an exponent in higher terms. Express, in like manner, $x^{\frac{1}{3}}$ and $y^{\frac{1}{5}}$.

3. By what equivalent expressions having fractional exponents with a common denominator, may $a^{\frac{1}{2}}$ and $a^{\frac{1}{3}}$ be expressed?

EXAMPLES.

1. Reduce $\sqrt{xy}$ and $\sqrt[3]{x^2y}$ to equivalent radicals of the same degree.

PROCESS.

$$\sqrt{xy} = (xy)^{\frac{1}{2}} = (xy)^{\frac{3}{6}} = \sqrt[6]{x^3y^3}$$

$$\sqrt[3]{x^2y} = (x^2y)^{\frac{1}{3}} = (x^2y)^{\frac{2}{6}} = \sqrt[6]{x^4y^2}$$

EXPLANATION.—Since the quantities are of different degrees, they may be reduced to the same degree by finding equivalent exponents which have for their denominator the least common multiple of the given denominators.

Expressing the quantities with their fractional exponents, they are $(xy)^{\frac{1}{2}}$ and $(x^2y)^{\frac{1}{3}}$. These, expressed as radicals of the same degree, are $(xy)^{\frac{3}{6}}$ and $(x^2y)^{\frac{2}{6}}$. Raising each quantity to the power indicated by the numerator of its exponent, and placing the result under the radical sign, with an index equal to the denominator of the exponent, we have as a result $\sqrt[6]{x^3y^3}$ and $\sqrt[6]{x^4y^2}$.

RULE.—*If necessary express the given radicals with fractional exponents. Reduce the fractional exponents to equivalent exponents having a common denominator.*

Raise each quantity to the power indicated by the numerator of its exponent, and indicate the root which is expressed by the common denominator.

Reduce the following to radicals of the same degree:

2. $\sqrt{10}$ and $\sqrt[3]{3}$.
3. $\sqrt{a^2x}$ and $\sqrt[3]{x^2y}$.
4. $\sqrt{x^2y^3}$ and $\sqrt[4]{xy^2}$.
5. $\sqrt[4]{a^2xy}$ and $\sqrt[3]{xy^2z}$.
6. $\sqrt[2]{axy^4}$ and $\sqrt[5]{a^2xy^2}$.
7. $\sqrt[n]{ax}$ and $\sqrt[m]{by}$.
8. $\sqrt{ax}$, $\sqrt[3]{by}$, and $\sqrt{az}$.
9. $\sqrt{a^2x}$, $\sqrt[3]{ay^2}$, and $\sqrt{by^3}$.
10. $\sqrt{\frac{3}{8}}$, $\sqrt[3]{\frac{3}{5}}$, and $\sqrt{2}$.
11. $\sqrt{\frac{x}{3}}$, $\sqrt[3]{\frac{x}{4}}$, and $\sqrt{x^3}$.
12. $\sqrt{a+b}$ and $\sqrt[3]{a+b}$.

ADDITION AND SUBTRACTION OF RADICALS.

248. PRINCIPLE.—*Only similar radicals can be combined in one term by addition or subtraction.*

1. Find the sum of $\sqrt{27}$, $2\sqrt{48}$, and $3\sqrt{108}$.

PROCESS.

$$\begin{aligned} \sqrt{27} &= 3\sqrt{3} \\ 2\sqrt{48} &= 8\sqrt{3} \\ 3\sqrt{108} &= 18\sqrt{3} \\ \hline \text{Sum} &= 29\sqrt{3} \end{aligned}$$

EXPLANATION.—Since the radical quantities are not similar, they must be reduced to similar radicals before adding. Expressing the radicals in their simplest form, $\sqrt{27} = 3\sqrt{3}$, $2\sqrt{48} = 8\sqrt{3}$, and $3\sqrt{108} = 18\sqrt{3}$. Therefore, since they all have the same radical factor with the same index, they are similar, and their sum is the sum of the coëfficients prefixed to the common radical factor.

RULE FOR ADDITION.—*Reduce the radicals, if necessary, to their simplest form. If the radicals are then similar, add their coëfficients and prefix the sum to the common radical factor; but if they are not then similar, indicate the process by connecting them with their proper signs.*

RULE FOR SUBTRACTION.—*Change the signs of the subtrahend, and proceed as in Addition.*

2. Find the sum of $\sqrt{18}$, $\sqrt{128}$, and $\sqrt{32}$.
3. Find the sum of $\sqrt{75}$, $\sqrt{12}$, and $\sqrt{48}$.
4. Find the sum of $\sqrt[3]{32}$, $\sqrt[3]{108}$, and $\sqrt[3]{256}$.
5. Find the sum of $3\sqrt{200}$, $4\sqrt{50}$, and $\sqrt{72}$.
6. Find the sum of $\sqrt[3]{162}$, $\sqrt[3]{384}$, and $\sqrt[3]{750}$.
7. Find the sum of $3\sqrt{\frac{2}{8}}$ and $7\sqrt{\frac{27}{50}}$.

8. Find the sum of $\sqrt{24}$, $\sqrt{54}$, and $\sqrt{294}$.

9. Find the sum of $\sqrt{\frac{3}{4}}$, $\sqrt{\frac{1}{3}}$, and $\frac{1}{2}\sqrt{3}$.

10. Find the sum of $\frac{1}{2}\sqrt{\frac{1}{2}}$, $\frac{3}{4}\sqrt{2}$, and $\sqrt{\frac{1}{8}}$.

11. Find the sum of $3\sqrt{242x^5y^5}$ and $11xy\sqrt{2x^3y^3}$.

12. From $\sqrt{243}$ take $\sqrt{108}$.

13. From $\sqrt{128x^2y^2}$ take $\sqrt{98x^2y^2}$.

14. From $\sqrt[3]{40}$ take $\sqrt[3]{135}$.

15. Simplify $\sqrt{24} + \sqrt{54} - \sqrt{96}$.

16. Simplify $\sqrt{12} + 3\sqrt{75} - 2\sqrt{27}$.

17. Simplify $5\sqrt{72} + 3\sqrt{18} - \sqrt{50}$.

18. Simplify $\sqrt{45a^3} + 5\sqrt{20a^3} - \sqrt{80a^3}$.

19. Simplify $\sqrt{8a^2y} + \sqrt{8b^2y} - \sqrt{50c^2y}$.

20. Simplify $\sqrt{2x^2 + 4xy + 2y^2} - \sqrt{2x^2 - 4xy + 2y^2}$.

21. Simplify $\sqrt[4]{32x^4y^3} + \sqrt[4]{162x^4y^3} - \sqrt[4]{512x^4y^3} + \sqrt[4]{1250x^4y^3}$.

22. Simplify $3\sqrt{\frac{3a}{4b^2}} + 2\sqrt{\frac{a}{3b^2}} - \sqrt{\frac{a}{27b^2}}$.

MULTIPLICATION OF RADICALS.

249. 1. When quantities have fractional exponents, what does the numerator of the exponent show? What the denominator?

2. Express with the radical sign $a^{\frac{2}{3}}$; $a^{\frac{5}{6}}$; $a^{\frac{3}{5}}$; $(ab)^{\frac{2}{3}}$; $(ab)^{\frac{3}{5}}$; $(a^2b)^{\frac{2}{3}}$.

3. Express with the radical sign $(6a)^{\frac{2}{3}}$; $(3a)^{\frac{3}{4}}$; $(5a)^{\frac{2}{5}}$; $(6a)^{\frac{4}{3}}$; $(6a^2b^3)^{\frac{4}{3}}$.

4. How is a^2 multiplied by a^3? a^5 by a^6? a^n by a^n? $a^{\frac{1}{3}}$ by $a^{\frac{1}{3}}$? $a^{\frac{3}{5}}$ by $a^{\frac{2}{5}}$? $a^{\frac{3}{7}}$ by $a^{\frac{2}{7}}$?

5. What is the product of $a^{\frac{2}{9}}$ by $a^{\frac{2}{9}}$? Of $a^{\frac{3}{8}}$ by $a^{\frac{5}{8}}$? Of $a^{\frac{1}{2}} \times a^{\frac{1}{4}}$? Of $a^{\frac{1}{3}} \times a^{\frac{1}{2}}$?

6. What must be done to the exponents $\frac{1}{2}$ and $\frac{1}{4}$ before they can be added?

7. When quantities have fractional exponents with different denominators, what must be done to the fractional exponents before multiplying?

EXAMPLES.

1. Multiply $3a\sqrt{6xy}$ by $2b\sqrt{3y}$.

PROCESS.

$$3a\sqrt{6xy} \times 2b\sqrt{3y} = 6ab\sqrt{18xy^2} =$$
$$6ab\sqrt{9y^2} \times \sqrt{2x} = 18aby\sqrt{2x}$$

EXPLANATION.—Since the radical quantities have the same index, their product may be found by multiplying together the various factors. Multiplying the coëfficients, we obtain $6ab$; multiplying the radical parts, we obtain $\sqrt{18xy^2}$. Consequently, the entire product is $6ab\sqrt{18xy^2}$, which, simplified, gives $18aby\sqrt{2x}$.

2. Multiply $\sqrt[3]{axy}$ by $\sqrt{ax^3y^3}$.

PROCESS.

$$\sqrt[3]{axy} \times \sqrt{ax^3y^3} = (axy)^{\frac{1}{3}} \times (ax^3y^3)^{\frac{1}{2}} = (axy)^{\frac{2}{6}} \times (ax^3y^3)^{\frac{3}{6}} =$$
$$\sqrt[6]{a^2x^2y^2} \times \sqrt[6]{a^3x^9y^9} = \sqrt[6]{a^5x^{11}y^{11}} = xy\sqrt[6]{a^5x^5y^5}$$

EXPLANATION.—Inasmuch as the quantities have not fractional exponents with the same denominator, they must be changed to equivalent quantities whose fractional exponents have the same denominator. Thus, they become $\sqrt[6]{a^2x^2y^2}$ and $\sqrt[6]{a^3x^9y^9}$. Multiplying as before, and simplifying, the result is $xy\sqrt[6]{a^5x^5y^5}$.

RULE.—*Reduce the radical factors to the same degree, if necessary.*

Multiply the coëfficients together for the coëfficient of the product, and the factors under the radical for the radical factor of the product, and simplify the result.

Multiply

3. $2\sqrt{8}$ by $3\sqrt{5}$.
4. $2\sqrt{12}$ by $3\sqrt{6}$.
5. $3\sqrt{9x}$ by $2\sqrt{3x}$.
6. $3\sqrt[3]{8y}$ by $2\sqrt[3]{8}$.
7. $6\sqrt[3]{4a}$ by $2\sqrt[3]{4c}$.
8. $5\sqrt{6xy}$ by $2\sqrt{3xy^2}$.
9. $4\sqrt[3]{2x^2y^2}$ by $5\sqrt[3]{4x^2y}$.
10. $5\sqrt[3]{32x^2y}$ by $6\sqrt[3]{4xy^3}$.
11. $2\sqrt{14}$ by $3\sqrt{20}$.

Multiply

12. $2\sqrt{15}$ by $3\sqrt{35}$.
13. $2\sqrt{50}$ by $\sqrt{200}$.
14. $\sqrt[3]{3a^2b}$ by $\sqrt[3]{9ab^2}$.
15. $\sqrt[3]{2x^2y}$ by $\sqrt{3x^3y^2}$.
16. $\sqrt[4]{a^2xy}$ by $\sqrt{a^2xy^2}$.
17. $\sqrt[3]{axy^2}$ by $\sqrt{3axy}$.
18. $2\sqrt{2xy}$ by $3\sqrt[3]{8xy^2}$.
19. $\sqrt{\frac{3x}{5}}$ by $\sqrt{\frac{4x}{7}}$.

250. The product of polynomials containing radical quantities, or quantities with fractional exponents, may be found by the rules already given.

20. What is the product of $2+\sqrt{6}$ multiplied by $2-\sqrt{6}$? Of $a^{\frac{1}{2}}+b^{\frac{1}{3}}$ multiplied by $a^{\frac{1}{4}}+b^{\frac{1}{2}}$?

PROCESS.

$$\begin{array}{r}
2+\sqrt{6} \\
2-\sqrt{6} \\ \hline
4+2\sqrt{6} \qquad\quad \\
-2\sqrt{6}-6 \\ \hline
4-6 \text{ or } -2
\end{array}$$

PROCESS.

$$\begin{array}{l}
a^{\frac{1}{2}}+b^{\frac{1}{3}} \\
a^{\frac{1}{4}}+b^{\frac{1}{2}} \\ \hline
a^{\frac{3}{4}}+a^{\frac{1}{4}}b^{\frac{1}{3}} \\
\qquad\qquad\quad +a^{\frac{1}{2}}b^{\frac{1}{2}}+b^{\frac{5}{6}} \\ \hline
a^{\frac{3}{4}}+a^{\frac{1}{4}}b^{\frac{1}{3}}+a^{\frac{1}{2}}b^{\frac{1}{2}}+b^{\frac{5}{6}}
\end{array}$$

21. Multiply $\sqrt{3} + \sqrt{2}$ by $\sqrt{3} - \sqrt{2}$.

22. Multiply $7 + \sqrt{3}$ by $7 - \sqrt{3}$.

23. Multiply $5 - \sqrt{2}$ by $5 - \sqrt{2}$.

24. Multiply $2x - \sqrt{y}$ by $2x + \sqrt{y}$.

25. Multiply $\sqrt{2} - \sqrt{15}$ by $\sqrt{3} - \sqrt{5}$.

26. Multiply $\sqrt{x} + \sqrt{y}$ by $\sqrt{x} - \sqrt{y}$.

27. Multiply $\sqrt{x} - \sqrt{y}$ by $\sqrt{x} - \sqrt{y}$.

28. Multiply $x + \sqrt{xy} + y$ by $x - \sqrt{xy} + y$.

29. Multiply $a - \sqrt{a^2 + b}$ by $a + \sqrt{a^2 + b}$.

30. Multiply $a^{\frac{2}{3}} + b^{\frac{2}{3}}$ by $a^{\frac{2}{3}} + b^{\frac{2}{3}}$.

31. Multiply $a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}$ by $a^{\frac{1}{3}} - b^{\frac{1}{3}}$.

32. Multiply $a^{\frac{3}{4}} - b^{\frac{3}{4}}$ by $a^{\frac{1}{4}} - b^{\frac{1}{4}}$.

33. Multiply $a^2 - a\sqrt{2} + 4$ by $a^2 + a\sqrt{2} + 4$.

34. Multiply $3 + 6\sqrt{5}$ by $4 + 5\sqrt{8}$.

DIVISION OF RADICALS.

251. 1. How is a^8 divided by a^2? a^6 by a^3? a^m by a^n? $a^{\frac{2}{3}}$ by $a^{\frac{1}{3}}$? $a^{\frac{3}{7}}$ by $a^{\frac{2}{7}}$? $a^{\frac{5}{9}}$ by $a^{\frac{3}{9}}$?

2. How are literal quantities divided?

3. What is the quotient of $a^{\frac{3}{7}} \div a^{\frac{2}{7}}$? Of $a^{\frac{5}{8}} \div a^{\frac{7}{8}}$? Of $a^{\frac{2}{5}} \div a^{\frac{1}{5}}$? Of $a^{\frac{1}{3}} \div a^{\frac{1}{2}}$? Of $a^{\frac{1}{2}} \div a^{\frac{1}{4}}$?

4. Since, in dividing literal quantities, the exponent of the divisor is subtracted from the exponent of the dividend, when quantities have fractional exponents with different denominators, what must be done to the exponents before dividing?

EXAMPLES.

1. Divide $3\sqrt{45}$ by $2\sqrt{9}$.

PROCESS.

$$3\sqrt{45} \div 2\sqrt{9} = 1\tfrac{1}{2}\sqrt{5}$$

EXPLANATION.—Since the radical quantities have the same index, the quotient may be found by dividing the various factors of the dividend by the corresponding factors of the divisor. Dividing the coëfficients, the quotient is $1\frac{1}{2}$; dividing the radical factors, the quotient is $\sqrt{5}$. Consequently, the entire quotient is $1\frac{1}{2}\sqrt{5}$.

2. Divide $\sqrt{ax^3y^3}$ by $\sqrt[3]{axy}$.

PROCESS.

$$\sqrt{ax^3y^3} \div \sqrt[3]{axy} = (ax^3y^3)^{\frac{1}{2}} \div (axy)^{\frac{1}{3}} =$$
$$(ax^3y^3)^{\frac{3}{6}} \div (axy)^{\frac{2}{6}} = \sqrt[6]{a^3x^9y^9} \div \sqrt[6]{a^2x^2y^2} =$$
$$\sqrt[6]{ax^7y^7} = xy\sqrt[6]{axy}$$

EXPLANATION.—Inasmuch as the quantities have not fractional exponents with the same denominator, they must be changed to equivalent quantities whose fractional exponents have the same denominator. Thus, they become $\sqrt[6]{a^3x^9y^9}$ and $\sqrt[6]{a^2x^2y^2}$. Dividing, as before, and simplifying, the result is $xy\sqrt[6]{axy}$.

RULE.—*Reduce the radical factors to the same degree, if necessary. Divide the coëfficient of the dividend by the coëfficient of the divisor, and the radical factors of the dividend by the radical factors of the divisor. The result will be the quotient.*

Divide

3. $6\sqrt{54}$ by $3\sqrt{2}$.
4. $2\sqrt{96}$ by $3\sqrt{6}$.
5. $4\sqrt{12a}$ by $2\sqrt{6}$.
6. $6\sqrt{12x^2}$ by $3\sqrt{4}$.

Divide

7. $\sqrt{96a^2}$ by $4a$.
8. $8\sqrt[3]{512}$ by $4\sqrt[3]{2}$.
9. $\frac{1}{2}\sqrt{5}$ by $\frac{1}{3}\sqrt{2}$.
10. $6\sqrt[3]{ax}$ by $2\sqrt{xy}$.

11. Divide $a\sqrt{\frac{x-y}{x+y}}$ by $b\sqrt{\frac{x-y}{x+y}}$.

12. Divide $\sqrt{x^2-y^2}$ by $x-y$.

13. Divide $\frac{1}{2}\sqrt[3]{\frac{1}{2}}$ by $\frac{1}{3}\sqrt[3]{\frac{1}{3}}$.

14. Divide $(4x^2)^{\frac{1}{3}}$ by $(2ax)^{\frac{1}{2}}$.

15. Divide $\sqrt{3x^2y-3xy^2}$ by $\sqrt{x-y}$.

16. Divide $4(x^2-y^2)^{\frac{1}{2}}$ by $2(x-y)^{\frac{1}{3}}$.

252. Polynomials containing radical quantities, or quantities having fractional exponents, may be divided according to the rules already given.

17. Divide $x^2+x\sqrt{y}-6y$ by $x-2\sqrt{y}$; also, $x^{\frac{1}{2}}+2x^{\frac{1}{4}}y^{\frac{1}{4}}+y^{\frac{1}{2}}$ by $x^{\frac{1}{4}}+y^{\frac{1}{4}}$.

PROCESS.

$$
\begin{array}{r|l}
x^2+x\sqrt{y}-6y & x-2\sqrt{y} \\ \cline{2-2}
x^2-2x\sqrt{y} & x+3\sqrt{y} \\ \cline{1-1}
3x\sqrt{y}-6y & \\
3x\sqrt{y}-6y &
\end{array}
$$

PROCESS.

$$
\begin{array}{r|l}
x^{\frac{1}{2}}+2x^{\frac{1}{4}}y^{\frac{1}{4}}+y^{\frac{1}{2}} & x^{\frac{1}{4}}+y^{\frac{1}{4}} \\ \cline{2-2}
x^{\frac{1}{2}}+x^{\frac{1}{4}}y^{\frac{1}{4}} & x^{\frac{1}{4}}+y^{\frac{1}{4}} \\ \cline{1-1}
x^{\frac{1}{4}}y^{\frac{1}{4}}+y^{\frac{1}{2}} & \\
x^{\frac{1}{4}}y^{\frac{1}{4}}+y^{\frac{1}{2}} &
\end{array}
$$

18. Divide $a^{\frac{1}{2}}-y^{\frac{1}{2}}$ by $a^{\frac{1}{4}}-y^{\frac{1}{4}}$.

19. Divide $a^{\frac{1}{2}}-2a^{\frac{1}{4}}y^{\frac{1}{4}}+y^{\frac{1}{2}}$ by $a^{\frac{1}{4}}-y^{\frac{1}{4}}$.

20. Divide $\sqrt{20}+\sqrt{12}$ by $\sqrt{5}+\sqrt{3}$.

21. Divide $6+11\sqrt{2}+6$ by $3+\sqrt{2}$.

22. Divide $8+10\sqrt{5}+15$ by $2+\sqrt{5}$.

23. Divide $12+17\sqrt{6}+36$ by $3+2\sqrt{6}$.

INVOLUTION OF RADICALS.

253. 1. What is the cube of $3\sqrt{x}$?

PROCESS.

$$(3\sqrt{x})^3 = (3x^{\frac{1}{2}})^3 =$$
$$3^3x^{\frac{3}{2}} = 27x^{\frac{3}{2}} =$$
$$27\sqrt{x^3} = 27x\sqrt{x}$$

EXPLANATION.—Expressing the radical quantity with a fractional exponent, and raising it to the third power, we have $3^3 \times x^{\frac{3}{2}}$, or $27x^{\frac{3}{2}}$. Expressing this in the form of a radical, and simplifying, the result is $27x\sqrt{x}$.

2. What is the square of $2 + 3\sqrt{x}$?

PROCESS.

$$(2 + 3\sqrt{x})^2 = 4 + 12\sqrt{x} + 9x$$

EXPLANATION.—Since the second power of the polynomial is sought, the power will be the square of the first term, plus twice the product of the first and second, plus the square of the second.

RULE.—*Raise the rational factor of a monomial radical to the required power, and annex to this he required power of the radical part, expressed under the radical sign or with fractional exponents.*

Expand a polynomial which contains a radical part according to the binomial formula, or perform the involution by actual multiplication.

3. Find the square of $3\sqrt{x}$.
4. Find the square of $2\sqrt[3]{2x^2}$.
5. Find the square of $4a\sqrt[3]{3b^2}$.
6. Find the cube of $3\sqrt{4x}$.
7. Find the cube of $2a\sqrt{3a}$.
8. Find the cube of $3\sqrt[3]{12ax}$.
9. Find the cube of $(a + b)^{\frac{1}{3}}$.

10. Find the cube of $(2a+3z)^{\frac{2}{3}}$.
11. Find the square of $\sqrt{3}+2\sqrt{5}$.
12. Find the square of $2+4\sqrt{3}$.
13. Find the square of $2+3\sqrt{5}$.
14. Find the square of $x^{\frac{1}{2}}+y^{\frac{2}{3}}$.
15. Find the square of $a^{\frac{3}{4}}+y^{\frac{2}{3}}$.

EVOLUTION OF RADICALS.

254. 1. What is the cube root of a^6? Of a^9? Of x^3y^6?

2. What is done to the exponent of a literal quantity to extract the cube root of the quantity? To extract the fourth root? To extract the fifth root? To extract any root?

EXAMPLES.

1. What is the cube root of $8a^3\sqrt{ab}$?

PROCESS.

$$8a^3\sqrt{ab}=8a^3a^{\frac{1}{2}}b^{\frac{1}{2}}$$

$$\sqrt[3]{8a^3a^{\frac{1}{2}}b^{\frac{1}{2}}}=2aa^{\frac{1}{6}}b^{\frac{1}{6}}=2a\sqrt[6]{ab}$$

OR

$$\sqrt[3]{8a^3\sqrt{ab}}=2a\sqrt[6]{ab}$$

EXPLANATION.—Since any root of a literal quantity may be found by dividing the exponent of the literal quantity by the index of the root sought, we express all the literal quantities with their exponents, obtaining $8a^3a^{\frac{1}{2}}b^{\frac{1}{2}}$, and divide the exponents by 3, and extract the cube root of the numerical factor. The result is $2aa^{\frac{1}{6}}b^{\frac{1}{6}}$, which, expressed as a radical, is $2a\sqrt[6]{ab}$.

The same result might have been obtained by extracting the cube root of the coëfficient and multiplying the index of the radical by 3.

RULE.—*Extract the required root of the coëfficient and of the quantity under the radical sign, by extracting the root of the numerical quantities, when possible, and dividing the exponents of the literal quantities by the index of the required root.* Or,

Extract the required root of the coëfficient, multiply the index of the radical by the index of the required root, and leave the quantity under the radical sign unchanged.

2. Find the square root of $16x^2\sqrt{az}$.
3. Find the square root of $36a^2x^2\sqrt{yz}$.
4. Find the cube root of $27a^3b^3\sqrt{xyz}$.
5. Find the fourth root of $a^8b^8\sqrt[3]{a^2x^2}$.
6. Find the cube root of $\frac{a}{3}\sqrt{\frac{a}{3}}$.
7. Find the square root of $16x\sqrt[3]{2y}$.
8. Find the square root of $(x+y)\sqrt{x+y}$.

RATIONALIZATION.

CASE I.

255. To rationalize a monomial surd.

1. By what quantity may $\sqrt{a}$ be multiplied to produce a rational quantity? $\sqrt[3]{a}$? $\sqrt{2a}$? $\sqrt[3]{a^2}$?

2. By what quantity may $a^{\frac{1}{2}}$ be multiplied to produce a rational quantity or a quantity with an integral exponent? $a^{\frac{2}{3}}$? $x^{\frac{1}{4}}$? $x^{\frac{2}{3}}$? $x^{\frac{3}{5}}$? $y^{\frac{2}{7}}$?

256. Rationalization is the process of removing the radical sign, or fractional exponent, from a quantity.

EXAMPLES.

1. Rationalize $\sqrt{2a}$.

PROCESS.

$$\sqrt{2a} \times \sqrt{2a} = 2a$$

Therefore, $\sqrt{2a}$ is the rationalizing factor.

EXPLANATION.—Since the square root of any quantity multiplied by itself will remove the radical sign, we multiply $\sqrt{2a}$ by $\sqrt{2a}$.

2. Rationalize $a^{\frac{1}{3}}x^{\frac{1}{2}}$.

PROCESS.

$$a^{\frac{1}{3}}x^{\frac{1}{2}} \times a^{\frac{2}{3}}x^{\frac{1}{2}} = ax$$

Therefore, $a^{\frac{2}{3}}x^{\frac{1}{2}}$ is the rationalizing factor.

EXPLANATION.—Since a radical quantity is rationalized by removing the radical sign or fractional exponent, we multiply $a^{\frac{1}{3}}x^{\frac{1}{2}}$ by such a factor as will make the exponents integral. Hence, to produce integral exponents in the product, we must multiply by a factor such that, when the exponent of each letter of the factor is added to the exponent of the corresponding letter of the surd, the sum of the exponents will be 1. Therefore, $a^{\frac{2}{3}}x^{\frac{1}{2}}$ is the rationalizing factor.

RULE.—*Multiply the surd by the same quantity with an exponent such that, when added to the fractional exponent of the surd, the sum of the exponents will be equal to 1.*

3. What factor will rationalize $\sqrt{4ax}$?
4. What factor will rationalize $\sqrt[3]{3a^2y}$?
5. What factor will rationalize $a\sqrt{3x}$?
6. What factor will rationalize $2\sqrt[3]{3y^2}$?
7. What factor will rationalize $2\sqrt[4]{aby}$?
8. What factor will rationalize $2ax\sqrt{3yz}$?
9. What factor will rationalize $4x^2y\sqrt[3]{3a^2y}$?

CASE II.

257. To rationalize a binomial surd of the second degree.

1. When $a+b$ is multiplied by $a-b$, what is the product?

2. When $\sqrt{a}+\sqrt{b}$ is multiplied by $\sqrt{a}-\sqrt{b}$, what is the product?

3. When $\sqrt{x}+\sqrt{y}$ is multiplied by $\sqrt{x}-\sqrt{y}$, what is the product?

EXAMPLES.

1. Rationalize $\sqrt{2}-\sqrt{3}$.

PROCESS.

$$\begin{array}{l} \sqrt{2}-\sqrt{3} \\ \sqrt{2}+\sqrt{3} \\ \hline 2-3=-1 \end{array}$$

EXPLANATION.—Inasmuch as each of the terms may be rationalized by squaring it, we may obtain the square of the terms by multiplying the binomial by the sum of the quantities. Hence, $\sqrt{2}+\sqrt{3}$ is the rationalizing factor.

RULE.—*Multiply the binomial by another binomial having the same quantities connected with the opposite sign.*

2. What factor will rationalize $\sqrt{5}-\sqrt{2}$?
3. What factor will rationalize $\sqrt{9}-\sqrt{6}$?
4. What factor will rationalize $x+\sqrt{3}$?
5. What factor will rationalize $x-3\sqrt{6}$?
6. What factor will rationalize $\sqrt{a}-\sqrt{x}$?
7. What factor will rationalize $2\sqrt{a}+3\sqrt{y}$?
8. What factor will rationalize $\sqrt{4a}-\sqrt{3x}$?
9. What factor will rationalize $x-\sqrt{y}$?
10. What factor will rationalize $x^2-\sqrt{yz}$?

CASE III.

258. To rationalize either term of a fraction.

1. Reduce $\frac{2a}{\sqrt{y}}$ to a fraction whose denominator is rational.

PROCESS.

$$\frac{2a}{\sqrt{y}} = \frac{2a \times \sqrt{y}}{\sqrt{y} \times \sqrt{y}} = \frac{2a\sqrt{y}}{y}$$

EXPLANATION.—Since the denominator is to be rationalized, we multiply the terms of the fraction by a quantity which will render the denominator a rational quantity. By Case I it is seen to be $\sqrt{y}$. Therefore, the fraction, when its denominator is rationalized, is $\frac{2a\sqrt{y}}{y}$.

RULE.—*Multiply the terms of the fraction by such a quantity as will render the required term rational.*

2. Rationalize the denominator of $\frac{3}{\sqrt{5}}$.

3. Rationalize the denominator of $\frac{2}{\sqrt{7}}$.

4. Rationalize the denominator of $\frac{4}{\sqrt{a}}$.

5. Rationalize the numerator of $\frac{2\sqrt{3}}{\sqrt{5}}$.

6. Rationalize the numerator of $\frac{2\sqrt{a}}{\sqrt{x}}$.

7. Rationalize the numerator of $\frac{2\sqrt{5}}{3\sqrt{7}}$.

8. Rationalize the denominator of $\dfrac{2}{\sqrt{2}-\sqrt{3}}$.

9. Rationalize the denominator of $\dfrac{2x}{\sqrt{a}+\sqrt{b}}$.

10. Rationalize the denominator of $\dfrac{2\sqrt{x}}{x-\sqrt{y}}$.

11. Rationalize the denominator of $\dfrac{2ab}{\sqrt{x}-\sqrt{y}}$.

12. Rationalize the denominator of $\dfrac{3}{\sqrt{x-1}-\sqrt{x+1}}$.

IMAGINARY QUANTITIES.

259. 1. What is the square root of a^2? Of a^4? Of $-a^2$? Of $-a^4$?

2. Into what factors may $\sqrt{-a^2}$ be separated so that one of them shall be a perfect square?

3. Into what factors may $\sqrt{-4}$ be separated so that one of them shall be a perfect square? $\sqrt{-9}$? $\sqrt{-16}$?

4. When the $\sqrt{a}$ is multiplied by the $\sqrt{a}$, what is the product? What, when $\sqrt{2x}$ is multiplied by $\sqrt{2x}$? $\sqrt{3y}$ by $\sqrt{3y}$?

5. What is the square of any radical quantity of the second degree? What is the square of $\sqrt{5}$? Of $\sqrt{-5}$? Of $\sqrt{-3a}$? Of $\sqrt{-5a}$?

260. An **Imaginary Quantity** is an indicated even root of a negative quantity.

Thus, $\sqrt{-2a}$, $\sqrt[4]{-3x}$, $\sqrt[6]{-a}$, are imaginary quantities.

261. Principle.—*Every imaginary monomial can be reduced to the form of* $a\sqrt[2n]{-1}$.

262. To add or subtract imaginary quantities.

1. Add $\sqrt{-x^2}$ and $2\sqrt{-4x^2}$.

PROCESS.

$$\begin{aligned} \sqrt{-x^2} &= x\sqrt{-1} \\ 2\sqrt{-4x^2} &= 4x\sqrt{-1} \\ &\overline{5x\sqrt{-1}} \end{aligned}$$

Explanation.—Since the radical expressions are dissimilar, they must be reduced to similar radicals before adding. Reducing and adding coëfficients, the sum is $5x\sqrt{-1}$.

2. Add $\sqrt{-a^2}$ and $\sqrt{-4a^2}$.
3. Add $2\sqrt{-4b^2}$ and $2b\sqrt{-9}$.
4. Add $3\sqrt{-16a^2x^2}$ and $a\sqrt{-25x^2}$.
5. From $\sqrt{-9a^2}$ subtract $a\sqrt{-16}$.
6. From $\sqrt{-3m^2x^2}$ subtract $\sqrt{-27m^2x^2}$.

263. To multiply imaginary quantities.

1. Multiply $2\sqrt{-3}$ by $2\sqrt{-6}$.

$$2\sqrt{-3} = 2\sqrt{3} \times \sqrt{-1}$$
$$2\sqrt{-6} = 2\sqrt{6} \times \sqrt{-1}$$
$$(2\sqrt{3} \times \sqrt{-1}) \times (2\sqrt{6} \times \sqrt{-1}) =$$
$$4\sqrt{18} \times (\sqrt{-1})^2 = -4\sqrt{18} = -12\sqrt{2}$$

Explanation.—In order to determine the sign of the product, we separate the imaginary quantities into their surd and imaginary factors. We then multiply, as in radical quantities, observing that $(\sqrt{-1})^2 = -1$. $\sqrt{-1} \times \sqrt{-1}$ would, according to the ordinary rules for multiplication of radicals, give as a product $\sqrt{+1}$, which is equal to ± 1; but, inasmuch as we know that $\sqrt{+1}$ was derived from the product of two negative factors, viz., $\sqrt{-1}$ and $\sqrt{-1}$, the root of $\sqrt{+1}$, in this case, is -1, and not $+1$. Hence,

264. PRINCIPLE.—*The product of two imaginary quantities is real, and the sign before the radical is determined by the ordinary rules reversed.*

2. Multiply $\sqrt{-4}$ by $\sqrt{-5}$.
3. Multiply $2\sqrt{-6}$ by $3\sqrt{-4}$.
4. Multiply $2\sqrt{-4}$ by $3\sqrt{-9}$.
5. Multiply $3\sqrt{-a^2}$ by $2\sqrt{-a^2x}$.
6. Multiply $1+\sqrt{-1}$ by $1-\sqrt{-1}$.
7. Multiply $1-\sqrt{-1}$ by $1-\sqrt{-1}$.

265. To divide imaginary quantities.

1. Divide $6\sqrt{-6}$ by $3\sqrt{-2}$.

PROCESS.

$$6\sqrt{-6}\div 3\sqrt{-2}=2\sqrt{3}$$

EXPLANATION.—Division of imaginary quantities is performed in the same manner as division of radicals.

2. Divide $3\sqrt{-8}$ by $\sqrt{-4}$.
3. Divide $2\sqrt{-a^2}$ by $3\sqrt{-a^2}$.
4. Divide $2\sqrt{-a^2x}$ by $3\sqrt{-x}$.
5. Divide 2 by $1+\sqrt{-1}$.
6. Divide $-2\sqrt{-1}$ by $1-\sqrt{-1}$.

REVIEW EXERCISES.

266. 1. Raise ax^2y^2 to the $m-1$ power.

2. Expand $(a+b)^7$.
3. Expand $(2a-3b)^4$.
4. Expand $\left(\frac{a}{2}-\frac{b}{3}\right)^3$.

5. Expand $\left(\frac{x}{4}+\frac{y}{3}\right)^5$.

6. Expand $\left(\frac{2x}{5}+\frac{3y}{7}\right)^8$.

7. Expand $(a+b)^n$ to 6 terms.

8. Expand $(a+b)^{n-2}$ to 5 terms.

9. Expand $(x+y)^r$ to 6 terms.

10. Expand $(1+2x+3y+z)^2$.

11. Extract the square root of $9x-24x^{\frac{1}{2}}y^{\frac{2}{3}}+12x^{\frac{1}{2}}+16y^{\frac{4}{3}}-16y^{\frac{2}{3}}+4$.

12. Extract the cube root of $5x^3-1-3x^5+x^6-3x$.

13. Divide m^2+mn+n^2 by $m+\sqrt{mn}+n$.

14. Divide x^4+y^4 by $x^2+xy\sqrt{2}+y^2$.

15. Multiply $\sqrt{a}+\sqrt{a-x}+\sqrt{x}$ by $\sqrt{a}-\sqrt{a-x}+\sqrt{x}$.

16. Multiply $\sqrt{a}+\sqrt{a-x}$ by $\sqrt{a}-\sqrt{a-x}$.

17. Cube $\sqrt[3]{2y^2}\times\sqrt[6]{x^3y}$.

18. Square $2+a^2\sqrt{2x+y}$.

19. Express the nth root of $\frac{a}{2}\sqrt{\frac{a}{2}}$.

20. Rationalize the denominator of $\frac{\sqrt{a}}{\sqrt{a}-\sqrt{b}}$.

21. Rationalize the denominator of $\frac{4+3\sqrt{2}}{3-2\sqrt{2}}$.

22. Rationalize the denominator of $\frac{a+\sqrt{a^2-x^2}}{a-\sqrt{a^2-x^2}}$.

23. Rationalize the denominator of $\frac{\sqrt{m^2+1}-\sqrt{m^2-1}}{\sqrt{m^2+1}+\sqrt{m^2-1}}$.

RADICAL EQUATIONS.

267. 1. In the equation $\sqrt{x}=2$, what is the value of x? How is it obtained?

2. In the equation $\sqrt{2x}=4$, how may the value of x be obtained? How in the equation $\sqrt{x}+2=4$?

3. What is the square of $2+\sqrt{x}$? What of $x+\sqrt{x}$? Which of the results contains the highest powers of x?

4. What is the square of $\sqrt{x}+\sqrt{2x}$? What of $\sqrt{x}+\sqrt{2}$? Which result contains the highest powers of x?

5. When the sum or difference of two radical expressions, one of which contains the unknown quantity in one term, and the other in both, is raised to a power corresponding to the index of the radical, which result contains the highest powers of the unknown quantity?

268. A **Radical Equation** is an equation containing a radical quantity.

Thus, $\sqrt{x}=5$, $\sqrt{x}+3=6$, $\sqrt{x}+5=\sqrt{3x}$, are radical equations.

EXAMPLES.

1. Given $\sqrt{x}+6=9$, to find the value of x.

PROCESS.

$$\sqrt{x}+6=9$$

Transposing and uniting, $\sqrt{x}=3$

Squaring, $x=9$

2. Given $\sqrt{4+x}=4-\sqrt{x}$, to find the value of x.

PROCESS.

$$\sqrt{4+x}=4-\sqrt{x}$$

Squaring, $\quad 4+x=16-8\sqrt{x}+x$

Transposing and uniting, $\quad 8\sqrt{x}=12$

Dividing by 4, $\quad 2\sqrt{x}=3$

Squaring, $\quad 4x=9$

$$x=2\tfrac{1}{4}$$

3. Given $\dfrac{x-ax}{\sqrt{x}}=\dfrac{\sqrt{x}}{x}$, to find the value of x.

PROCESS.

$$\frac{x-ax}{\sqrt{x}}=\frac{\sqrt{x}}{x}$$

Clearing of fractions, $\quad x^2-ax^2=x$

Dividing by x, $\quad x-ax=1$

Factoring, $\quad (1-a)x=1$

$$x=\frac{1}{1-a}$$

269. From an examination of the solutions of the above examples, some of the following suggestions will be seen to be of value.

Experience will teach the student the value of the other suggestions, and his own ingenuity will devise elegant methods, not mentioned here.

SUGGESTIONS.—1. *Transpose the terms so that the radical quantity, if there be but one,—or the more complex radical, if there be more than one,—may constitute one member of the*

equation; then raise each member to a power of the same degree as the radical.

2. *When the equation is not freed from radicals by the first involution, proceed again as indicated in Suggestion 1.*

3. *Simplify the equation as much as possible before performing the involution.*

4. *Sometimes it may be advantageous to clear a radical equation of fractions, either in whole or in part. Especially will it be so when a radical denominator and another numerator are similar.*

5. *It is sometimes convenient to rationalize the denominator before clearing of fractions or involving.*

4. Given $\sqrt{x}+7=9$, to find x.

5. Given $\sqrt{2x}-5=3$, to find x.

6. Given $\sqrt{x-7}=7$, to find x.

7. Given $\sqrt[3]{4x-16}=2$, to find x.

8. Given $\sqrt{x+9}=6$, to find x.

9. Given $\sqrt[3]{2x+3}+4=7$, to find x.

10. Given $\sqrt{x^2-9}+x=9$, to find x.

11. Given $\sqrt{x^2-11}+1=x$, to find x.

12. Given $\sqrt{16+x}+\sqrt{x}=8$, to find x.

13. Given $\sqrt{x-16}=8-\sqrt{x}$, to find x.

14. Given $\sqrt{x-21}=\sqrt{x}-1$, to find x.

15. Given $\sqrt{x}+\sqrt{x-9}=\dfrac{36}{\sqrt{x-9}}$, to find x.

16. Given $\sqrt{x}+\sqrt{x-4}=\dfrac{4}{\sqrt{x-4}}$, to find x.

17. Given $\dfrac{\sqrt{x}+16}{\sqrt{x}+4}=\dfrac{\sqrt{x}+32}{\sqrt{x}+12}$, to find x.

18. Given $\dfrac{3x-1}{\sqrt{3x}+1}-\dfrac{\sqrt{3x}-1}{2}=4$, to find x.

19. Given $x+\sqrt{x^2-\sqrt{1-x}}=1$, to find x.

20. Given $\sqrt{1+x\sqrt{x^2+12}}=1+x$, to find x.

21. Given $\dfrac{\sqrt{x}-8}{\sqrt{x}-6}=\dfrac{\sqrt{x}-4}{\sqrt{x}+2}$, to find x.

22. Given $\sqrt{a-x}=\dfrac{a}{\sqrt{a-x}}-x$, to find x.

23. Given $\dfrac{ax-1}{\sqrt{ax}+1}=\dfrac{\sqrt{ax}-1}{2}+4$, to find x.

24. Given $\dfrac{\sqrt{x}+28}{\sqrt{x}+4}=\dfrac{32}{\sqrt{x}+6}+1$, to find x.

25. Given $\dfrac{3\sqrt{2x}+10}{3\sqrt{2x}-10}=\dfrac{\sqrt{2x}+16}{\sqrt{2x}-4}$, to find x.

26. Given $\sqrt{x+8}-\sqrt{x-8}=2\sqrt{2}$, to find x.

27. Given $\dfrac{a}{x}+\dfrac{\sqrt{a^2-x^2}}{x}=\dfrac{1}{b}$, to find x.

28. Given $\sqrt{\dfrac{1}{x}}-\sqrt{x}=\sqrt{1+x}$, to find x.

29. Given $\dfrac{\sqrt{a+x}+\sqrt{x}}{\sqrt{a+x}-\sqrt{x}}=b$, to find x.

30. Given $\dfrac{\sqrt{4x+1}+2\sqrt{x}}{\sqrt{4x+1}-2\sqrt{x}}=9$, to find x.

31. Given $\dfrac{\sqrt{5+x}+\sqrt{5-x}}{\sqrt{5+x}-\sqrt{5-x}}=2$, to find x.

32. Given $\dfrac{\sqrt{a}+\sqrt{a-x}}{\sqrt{a}-\sqrt{a-x}}=\dfrac{1}{a}$, to find x.

33. Given $\dfrac{b-\sqrt{b^2-x}}{b+\sqrt{b^2-x}}=a$, to find x.

QUADRATIC EQUATIONS.

270. 1. How is the degree of an equation determined? (Art. 177.)

2. What is the degree of the equation $x+2=7$? Of $x^2+4=8$? Of $x^2+4x=15$? Of $x+xy=12$?

3. What are equations of the second degree called?

4. When $x^2=4$, what is the value of x? What, when $x^2=9$? What, when $x^2=a^2$?

5. How many values has x in these equations? How do the numerical values of x compare? What are the signs of the values of x?

DEFINITIONS.

271. A **Quadratic Equation** is an equation of the second degree.

Thus, $x^2=9$, and $x^2+bx=c$, are quadratic equations.

272. A **Pure Quadratic Equation** is an equation which contains only the second power of the unknown quantity.

Thus, $4x^2=16$, and $ax^2=b$, are pure quadratics.

273. A Pure Quadratic Equation is sometimes called an *Incomplete Quadratic Equation.*

274. A **Root** of an equation is the value of the unknown quantity.

PURE QUADRATICS.

275. Since pure quadratic equations contain only the second power of the unknown quantity, they may always be reduced to the general form of $ax^2 = b$, in which a represents the coëfficient of x^2, and b the other terms.

276. PRINCIPLE.—*Every pure quadratic equation has two roots numerically equal, but having opposite signs.*

EXAMPLES.

1. Given $3x^2 + \dfrac{x^2}{2} = 14$, to find the value of x.

PROCESS.

$$3x^2 + \frac{x^2}{2} = 14$$

Clearing of fractions, $6x^2 + x^2 = 28$

Uniting terms, $7x^2 = 28$

$x^2 = 4$

Extracting the square root, $x = \pm 2$

2. Given $ax^2 + c = bx^2 + d$, to find x.

PROCESS.

$$ax^2 + c = bx^2 + d$$

Transposing, $ax^2 - bx^2 = d - c$

Factoring, $(a - b)\,x^2 = d - c$

$$x^2 = \frac{d - c}{a - b}$$

Extracting the square root, $x = \pm\sqrt{\dfrac{d - c}{a - b}}$.

3. Given $x^2 - 3 = 46$, to find the value of x.

4. Given $3x^2 + 7 = x^2 + 15$, to find the value of x.

5. Given $2x^2 - 6 = 66$, to find the value of x.

6. Given $\frac{x^2 - 12}{3} = \frac{x^2 - 4}{4}$, to find the value of x.

7. Given $5x^2 - 3 = 2x^2 + 24$, to find the value of x.

8. Given $3x^2 + 7 = \frac{3x^2}{4} + 43$, to find the value of x.

9. Given $2x^2 = \frac{3x^2}{5} + 35$, to find the value of x.

10. Given $\frac{x^2 + 2}{3} + 8 = 42$, to find the value of x.

11. Given $\frac{x - 3}{4} = \frac{4}{x + 3}$, to find the value of x.

12. Given $\frac{x^2}{3} + 9 = \frac{3x^2 - 3}{5}$, to find the value of x.

13. Given $\frac{x + 4}{x - 4} + \frac{x - 4}{x + 4} = 3\frac{1}{3}$, to find the value of x.

14. Given $\frac{1}{1 - x} + \frac{1}{1 + x} = 2\frac{2}{3}$, to find the value of x.

15. Given $\frac{x^2 + 9x}{15} = \frac{3(x + 2)}{5}$, to find the value of x.

16. Given $(x + 2)(x - 2) = \frac{x^2}{4} - 1$, to find the value of x.

17. Given $\sqrt{a + x} = \frac{a}{\sqrt{x - a}}$, to find the value of x.

18. Given $\frac{15}{\sqrt{x^2 + 5}} = x + \sqrt{x^2 + 5}$, to find the value of x.

19. Given $x + \sqrt{x^2 - 2\sqrt{1 - x}} = 1$, to find the value of x.

20. Given $\sqrt{a+x}=\sqrt{x+\sqrt{x^2-b^2}}$, to find the value of x.

21. Given $\dfrac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^2+1}+\sqrt{x^2-1}}=\dfrac{1}{2}$, to find the value of x.

22. Given $\dfrac{2}{x+\sqrt{2-x^2}}+\dfrac{2}{x-\sqrt{2-x^2}}=\dfrac{a}{x}$, to find the value of x.

23. Given $\dfrac{1+x}{1+x+\sqrt{1+x^2}}=a-\dfrac{1-x}{1-x+\sqrt{1+x^2}}$, to find the value of x.

PROBLEMS.

277. 1. What number is that, to the square of which, if $\frac{5}{9}$ be added, the sum will be 1?

2. Find a number such that the square of $\frac{3}{4}$ of it will be 7 less than the square of it.

3. Find a number such that if 320 is divided by it, and the quotient added to the number itself, the sum will be equal to 6 times the number.

4. If a certain number is increased by 3 and also diminished by 3, the product of the sum and difference will be 55. What is the number?

5. Two numbers are to each other as 3 to 5, and the sum of their squares is 3400. What are the numbers?

6. A gentleman said that his son's age was $\frac{1}{4}$ of his own age, and that the difference of the squares of the numbers which represent their ages was 960. What were their ages?

7. A man lent a sum of money at 6% per annum, and found that, if he multiplied the principal by the number which expressed the interest for 8 months, the product would be $900. What was the principal?

8. A gentleman has two square rooms whose sides are to each other as 2 to 3. He finds that it will require 20 square yards more of carpeting to cover the floor of the larger than of the smaller room. What is the length of one side of each room?

9. The sum of two numbers is 12, and their product is 27. What are the numbers?

NOTE.—Let $6 + x =$ one number, and $6 - x =$ the other number.

10. The sum of two numbers is 15, and their product is 56. What are the numbers?

11. The sum of two numbers is 13, and their product is 42. What are the numbers?

12. Divide 20 into two such parts that their product will be 96.

13. Divide 32 into two such parts that their product will be 240.

14. A merchant bought a piece of cloth for $24, paying $\frac{3}{8}$ as many dollars per yard as there were yards in the piece. How many yards were there?

15. A man purchased a rectangular field whose length was $1\frac{1}{9}$ times its breadth. It contained 9 acres. What was the length of each side?

16. Find two numbers which are to each other as 5 to 4, and the sum of whose squares is 164.

AFFECTED QUADRATICS.

278. 1. How is a binomial squared? What is the square of $x + 2$?

2. Since the second term of the square of a binomial contains twice the product of both terms, if the second term of the square is given, how may the second term of the *binomial* be found when the first term of the binomial is known?

3. What term is it necessary to add to $x^2 + 4x$ to make a perfect square? How is the term found?

4. What term must be added to $x^2 + 6x$ to make a perfect square? How is it found?

5. What must be added to $x^2 + 8x$ to make it a perfect square?

6. What is the square root of the completed square of which $x^2 + 8x$ are two terms?

7. What is the square root of the completed square of which $x^2 + 12x$ are two terms?

8. What is the square root of the completed square of which $x^2 + 6x$ are two terms?

9. What is the square root of the completed square of which $x^2 + 10x$ are two terms?

10. What is the square root of the completed square of which $x^2 + 20x$ are two terms?

11. In the equation $x^2 + 4x + 4 = 9$, what is the square root of the first member? What is the square root of the second member?

12. Since, in the solution of the equation $x^2 + 4x + 4 = 9$, we obtain the result $x + 2 = \pm 3$, how many values has x?

13. In the equation $x^2 + 6x + 9 = 16$, what is the square root of the first member? What of the second member? How many values has x?

DEFINITIONS.

279. An **Affected Quadratic Equation** is an equation which contains both the first and second powers of the unknown quantity.

Thus, $x^2 + 2x = 4$, $5x^2 + 6x = 8$, and $ax^2 + bx = c$, are affected quadratic equations.

280. An Affected Quadratic Equation is sometimes called a *Complete Quadratic Equation.*

281. Since affected quadratic equations contain both the second and first powers of the unknown quantity, they may always be reduced to the general form of $ax^2 + bx = c$, in which a and b represent respectively the coëfficients of x^2 and x, and c the other terms.

282. PRINCIPLE.—*Every affected quadratic equation has two roots, and only two.*

These roots are always numerically unequal, except when the second member of the equation reduces to 0.

283. First method of completing the square.

EXAMPLES.

1. Given $x^2 + 4x = 96$, to find the values of x.

PROCESS.

$$x^2 + 4x = 96$$
$$x^2 + 4x + 4 = 96 + 4$$
$$x^2 + 4x + 4 = 100$$
$$x + 2 = \pm 10$$
$$x = 10 - 2 = 8$$
$$x = -10 - 2 = -12$$

EXPLANATION.—Completing the square in the first member by adding the square of one-half the coëfficient of x to both members, we have $x^2 + 4x + 4 = 96 + 4$. Extracting the square root of each member, we have $x + 2 = \pm 10$. Using, first, the positive value of 10, we obtain 8 for one value of x; using next the negative value of 10, we obtain -12 for the other value of x.

VERIFICATION.

$$64 + 32 = 96 \quad (1)$$
$$144 - 48 = 96 \quad (2)$$

Substituting the values of x for x, in the original equation, we see that the values are correct.

2. Given $x^2 - 5x = 24$, to find the values of x.

PROCESS.

$$x^2 - 5x = 24$$

Completing the square, $x^2 - 5x + \frac{25}{4} = 24 + \frac{25}{4}$

Uniting terms in second member, $x^2 - 5x + \frac{25}{4} = \frac{121}{4}$

Extracting the square root, $x - \frac{5}{2} = \pm\frac{11}{2}$

Transposing, $x = \frac{5}{2} + \frac{11}{2} = 8$

$$x = \frac{5}{2} - \frac{11}{2} = -3$$

Therefore, $x = 8$ or -3

3. Given $2x^2 - 7x = 30$, to find the values of x.

PROCESS.

$$2x^2 - 7x = 30$$

Dividing by coëfficient of x^2, $x^2 - \frac{7}{2}x = 15$

Completing the square, $x^2 - \frac{7}{2}x + \frac{49}{16} = 15 + \frac{49}{16} = \frac{289}{16}$

Extracting the square root, $x - \frac{7}{4} = \pm\frac{17}{4}$

Transposing, $x = \frac{7}{4} + \frac{17}{4} = 6$

$$x = \frac{7}{4} - \frac{17}{4} = \frac{-10}{4} = -2\tfrac{1}{2}$$

Therefore, $x = 6$ or $-2\frac{1}{2}$

RULE.—*Reduce the equation to the form $x^2 \pm bx = \pm c$, by dividing both members of the equation by the coëfficient of the highest power of the unknown quantity.*

Add the square of one-half the coëfficient of the second term to both members of the equation, extract the square root of each member, and reduce the equation.

The solution of the equation $x^2 + bx = c$ gives $x = -\frac{b}{2} \pm \sqrt{c + \frac{b^2}{4}}$. Hence, the values of the unknown quantity may be found as follows:

Write one-half the coëfficient of the second term with the contrary sign, $\pm$ the square root of the sum of the square of half this coëfficient and the second member of the equation.

Inasmuch as it is impossible to extract the square root of a negative quantity, it is necessary that the term containing the second power of the unknown quantity should have the *positive* sign. If it should be negative, change the signs of all the terms in both members.

Find the values of x in the following equations:

4. $x^2 + 4x = 45$.
5. $x^2 + 6x = 27$.
6. $x^2 + 8x = 20$.
7. $x^2 + 10x = 11$.
8. $x^2 + 20x = 21$.
9. $x^2 + 18x = 19$.
10. $x^2 + 24x = 25$.
11. $x^2 - 12x = 45$.
12. $x^2 - 8x = 33$.
13. $x^2 - 14x = 51$.
14. $x^2 - 28x = 60$.
15. $x^2 - 30x = 64$.
16. $2x^2 + 3x = 14$.
17. $3x^2 + 4x = 39$.
18. $2x^2 + 7x = 39$.
19. $5x^2 + 15x = 50$.
20. $6x^2 - 21x = 12$.
21. $\frac{x-5}{10} = \frac{x+1}{x-6}$.
22. $\frac{x+2}{5} = \frac{8}{3x+4}$.
23. $3x - \frac{3x-3}{x-3} = \frac{3x-6}{2}$.

284. Other methods of completing the square.

When the second power of the unknown quantity has coëfficients, the following method, sometimes called the Hindoo Method, will avoid fractions.

Let it be required to find the values of x in the equation $ax^2 + bx = c$.

PROCESS.

$$ax^2 + bx = c \tag{1}$$

Dividing by the coëfficient of x^2, $x^2 + \frac{b}{a}x = \frac{c}{a}$ (2)

Completing the square, $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{c}{a} + \frac{b^2}{4a^2}$ (3)

Multiplying by $4a^2$, $4a^2x^2 + 4abx + b^2 = 4ac + b^2$ (4)

EXPLANATION.—Solving this partially by the preceding method, we find, when the square is completed, that equation (3) is obtained. By multiplying the members of this equation by $4a^2$, all fractions are removed, giving equation (4). That is, *in completing the square, fractions may be avoided by multiplying both members by 4 times the coëfficient of x^2, and adding the square of the coëfficient of x to both members.*

When the coëfficient of x is an even number, the method may be modified as follows:

Let it be required to find the values of x in the equation $ax^2 + 2dx = c$, in which $2d$ is an even number.

PROCESS.

$$ax^2 + 2dx = c \tag{1}$$

$$x^2 + \frac{2dx}{a} = \frac{c}{a} \tag{2}$$

$$x^2 + \frac{2d}{a}x + \frac{d^2}{a^2} = \frac{c}{a} + \frac{d^2}{a^2} \tag{3}$$

Multiplying by a^2, $a^2x^2 + 2adx + d^2 = ac + d^2$ (4)

EXPLANATION.—Solving partially by the first method, equation (4) is obtained after clearing of fractions. That is, *in completing the square, when the coëfficient of x is even, fractions may be avoided by multiplying both members of the equation by the coëfficient of x^2, and adding the square of one-half of the coëfficient of x to both members.*

EXAMPLES.

1. Given $2x^2 + 3x = 14$, to find the values of x.

PROCESS.

$$2x^2 + 3x = 14$$
$$16x^2 + 24x + 9 = 112 + 9$$
$$4x + 3 = \pm 11$$
$$4x = 8, \text{ or } -14$$
$$x = 2, \text{ or } -3\tfrac{1}{2}$$

EXPLANATION.—Inasmuch as x^2 has a coëfficient and the coëfficient of x is not even, to avoid fractions we multiply both members by 4 times this coëfficient, or 8, and add the square of the coëfficient of x, or 9, to both members.

Extracting the square root of each member, transposing, etc., the values of x are 3, and $-3\frac{1}{2}$.

2. Given $5x^2 + 4x = 57$, to find the values of x.

PROCESS.

$$5x^2 + 4x = 57$$
$$25x^2 + 20x + 4 = 275 + 4$$
$$5x + 2 = \pm 17$$
$$5x = 15, \text{ or } -19$$
$$x = 3, \text{ or } -3\tfrac{4}{5}$$

EXPLANATION.—Inasmuch as x^2 has a coëfficient, and the coëfficient of x is an even number, to avoid fractions we multiply both members of the equation by the coëfficient of x^2, which is 5, and add the square of one-half the coëfficient of x, or 4, to both members.

Extracting the square root of each member, transposing, etc., the values of x are 3, and $-3\frac{4}{5}$.

RULE.—*If the coëfficient of the first power of the unknown quantity be odd, multiply the equation by four times the coëfficient of the second power of the unknown quantity, and add the square of the coëfficient of the first power to both members.*

If the coëfficient of the first power of the unknown quantity be even, multiply the equation by the coëfficient of the second power, and add the square of one-half the coëfficient of the first power to both members.

When the coëfficient of the second power of the unknown quantity is a perfect square, divide the coëfficient of the first power by twice the square root of the coëfficient of the second power, and add the square of the result to both members.

Extract the square root of both members, and find the value of x in the resulting equation.

The student may solve, in a manner similar to the above, $a^2x^2 + bx = c$, and deduce a rule similar to that already given for completing the square when the coëfficient of x^2 is a square.

Find the values of x in the following:

3. $3x^2 + 5x = 8.$
4. $2x^2 + 7x = 22.$
5. $4x^2 + 5x = 84.$
6. $5x^2 - 4x = 105.$
7. $3x^2 - 16x = 140.$
8. $4x^2 - 7x = 102.$
9. $9x^2 + 4x = 44.$
10. $8x^2 - 6x = 464.$
11. $5x^2 - 6x = 144.$
12. $3x^2 + 2ax = b.$
13. $x^2 - 6x - 14 = 2.$
14. $x^2 - 13x - 6 = 8.$
15. $x^2 + 17x - 18 = 0.$
16. $x^2 - 11x - 7 = 5.$
17. $2x^2 - 18x = -40.$
18. $2x^2 + 5x = 18.$
19. $3x^2 + 2x = 21.$
20. $2x^2 - 7x = 34.$
21. $5x^2 - 6x = 41.$
22. $\frac{4x}{x+3} - \frac{x-3}{2x+5} = 2.$
23. $3x - \frac{169}{x} = 26.$
24. $\frac{7}{4} - \frac{2x-5}{x+5} = \frac{3x-7}{2x}.$
25. $\frac{3x-5}{9x} - \frac{6x}{3x-25} = \frac{1}{3}.$
26. $\frac{1}{x-1} - \frac{2}{x+2} = \frac{1}{2}.$
27. $\frac{x}{x+60} = \frac{7}{3x-5}.$
28. $\frac{x+11}{x} = 7 - \frac{9+4x}{x^2}.$
29. $\frac{4x-10}{x+5} - \frac{7-3x}{x} = \frac{7}{2}.$

EQUATIONS IN THE QUADRATIC FORM.

285. An equation which contains but two powers of an unknown quantity, the exponent of one power being twice that of the other power, is in the *Quadratic Form.*

These equations in the quadratic form can be reduced to the general form $ax^{2n}+bx^n=c$, in which n represents any number.

EXAMPLES.

1. Given $x^4+3x^2=28$, to find the values of x.

PROCESS.

$$x^4+3x^2=28$$

Completing the square, $x^4+3x^2+\frac{9}{4}=\frac{121}{4}$

Extracting the square root, $x^2+\frac{3}{2}=\pm\frac{11}{2}$

$$x^2=4 \text{ or } -7$$

Extracting the square root, $x=\pm 2 \text{ or } \pm\sqrt{-7}$

2. Given $x^{\frac{2}{3}}+3x^{\frac{1}{3}}=10$, to find the values of x.

FIRST PROCESS.

$$x^{\frac{2}{3}}+3x^{\frac{1}{3}}=10$$

Completing the square, $x^{\frac{2}{3}}+3x^{\frac{1}{3}}+\frac{9}{4}=\frac{49}{4}$

Extracting the square root, $x^{\frac{1}{3}}+\frac{3}{2}=\pm\frac{7}{2}$

$$x^{\frac{1}{3}}=2 \text{ or } -5$$

Raising to third power, $x=8 \text{ or } -125$

SECOND PROCESS.

$$\text{Let} \quad x^{\frac{1}{3}} = p$$

$$\text{Then,} \quad x^{\frac{2}{3}} = p^2$$

Substituting in giver equation, $p^2 + 3p = 10$

$$p^2 + 3p + \frac{9}{4} = \frac{49}{4}$$

$$p + \frac{3}{2} = \pm\frac{7}{2}$$

$$p = 2 \text{ or } -5$$

$$\text{Hence,} \quad x^{\frac{1}{3}} = 2 \text{ or } -5$$

$$\text{Cubing,} \quad x = 8 \text{ or } -125$$

3. Given $x^4 - 2x^2 = 8$, to find the values of x.
4. Given $x^6 - 3x^3 = 40$, to find the values of x.
5. Given $x^6 - 4x^3 = 32$, to find the values of x.
6. Given $2x^4 - 4x^2 = 16$, to find the values of x.
7. Given $2x^2 - 1 = \frac{3}{x^2}$, to find the values of x.
8. Given $x^4 + 4x^2 = 12$, to find the values of x.
9. Given $x^{\frac{1}{2}} + 2x^{\frac{1}{4}} = 8$, to find the values of x.
10. Given $x^3 + 3x^{\frac{3}{2}} = 88$, to find the values of x.
11. Given $x^{\frac{2}{3}} + 3x^{\frac{1}{3}} = 4$, to find the values of x.
12. Given $x^{\frac{1}{2}} - x^{\frac{1}{4}} = 20$, to find the values of x.
13. Given $ax^{2n} + bx^n = c$, to find the values of x.

286. Polynomials are sometimes affected with exponents, one of which is twice the other. When such expressions containing the unknown quantity are found in equations, the equations may be solved like the preceding.

14. Given $(x+2)^2+(x+2)=20$, to find the values of x.

FIRST PROCESS.

$$(x+2)^2+(x+2)=20$$

Completing the square, $(x+2)^2+(x+2)+\frac{1}{4}=\frac{81}{4}$

Extracting the square root, $\quad (x+2)+\frac{1}{2}=\pm\frac{9}{2}$

$$x+2=4 \text{ or } -5$$
$$x=2 \text{ or } -7$$

SECOND PROCESS.

Let $\quad p=(x+2)$

Then, $\quad p^2=(x+2)^2$

Then, $p^2+p=20$

$$p^2+p+\frac{1}{4}=\frac{81}{4}$$

$$p+\frac{1}{2}=\pm\frac{9}{2}$$

$$p=4 \text{ or } -5$$
$$x+2=4 \text{ or } -5$$
$$x=2 \text{ or } -7$$

15. Given $(x^2+1)^2+(x^2+1)=30$, to find the values of x.

16. Given $(x^2+4)^2+(x^2+4)=30$, to find the values of x.

17. Given $(x-1)^2+5(x-1)=14$, to find the values of x.

18. Given $(x^2-9)^2-11(x^2-9)=80$, to find the values of x.

19. Given $(x^2-x)^2-(x^2-x)=132$, to find the values of x.

20. Given $x+5-\sqrt{x+5}=6$, to find the values of x.

21. Given $3x+4+4\sqrt{3x+4}=32$, to find the values of x.

22. Given $\left(\frac{8}{x}+x\right)^2+\left(\frac{8}{x}+x\right)=42$, to find the values of x.

23. Given $\left(\frac{6}{x}+x\right)^2+\left(\frac{6}{x}+x\right)=30$, to find the values of x.

24. Given $(2x^2-4x+1)^2-(2x^2-4x+1)=42$, to find the values of x.

25. Given $x^2-7x+18+\sqrt{x^2-7x+18}=42$, to find the values of x.

26. Given $2x^2+3x+9-5\sqrt{2x^2+3x+9}=6$, to find the values of x.

27. Given $2(3x^2+1)^{\frac{1}{2}}+3x^2+1=63$, to find the values of x.

28. Given $\sqrt{x+12}+\sqrt[4]{x+12}=6$, to find the values of x.

PROBLEMS.

287. 1. Find two numbers whose sum is 12, and whose product is 35.

SOLUTION.

Let $x=$ one.

Then, $12-x=$ the other.

$$12x-x^2=35$$
$$x^2-12x=-35$$
$$x^2-12x+36=1$$
$$x-6=\pm 1$$
$$x=7 \text{ or } 5$$
$$12-x=5 \text{ or } 7$$

2. The sum of two numbers is 10, and their product is 21. What are the numbers?

3. Divide 27 into two such parts that their product may be 140.

4. A rectangular field is 12 rods longer than it is wide, and contains seven acres. What is the length of its sides?

5. A person purchased a flock of sheep for $100. If he had purchased 5 more for the same sum, they would have cost $1 less per head. How many did he buy?

6. An orchard containing 2000 trees had 10 rows more than it had trees in a row. How many rows were there? How many trees were there in each row?

7. The difference between two numbers is 2, and the sum of their squares is 244. What are the numbers?

8. One hundred and ten dollars was divided among a certain number of persons. If each person had received $1 more, he would have received as many dollars as there were persons. How many persons were there?

9. A man worked a certain number of days, receiving for his pay $18. If he had received $1 per day less than he did he would have had to work 3 days longer to earn the same sum. How many days did he work?

10. Find the price of eggs when 2 less for 12 cents raises the price 1 cent per dozen.

11. A person sold goods for $24, gaining a per cent. equal to the number of dollars which the goods cost him. What did they cost him?

Let x = the cost; then, $\frac{x}{100}$ = the gain per cent.

12. The expenses of a party amount to $10. If each pays 30 cents more than there are persons, the bill will be settled. How many are there in the party?

13. A picture, which is 18 inches by 12, is to be

surrounded with a frame of uniform width, whose area is equal to that of the glass. What is the width of the frame?

14. A man sold a quantity of goods for \$39, and gained a per cent. equal to the number of dollars which the goods cost him? What did they cost him?

15. Two men dig a ditch 100 rods in length for \$100, each receiving \$50. A is to have 25 cents a rod more than B. How many rods does each dig? What is the price per rod?

16. A rectangular park, 60 rods long and 40 rods wide, is surrounded by a street of uniform width, containing 1344 square rods. How wide is the street?

17. A person purchased two pieces of cloth which together measured 36 yards. Each cost as many shillings per yard as there were yards in the piece. If one piece cost 4 times as much as the other, how many yards were there in each?

18. A person drew a quantity of pure wine from a vessel which was full, holding 81 gallons, and then filled the vessel up with water. He then drew from the mixture as much as he drew before of pure wine, when it was found that the vessel contained 64 gallons of pure wine. How much did he draw each time?

19. Two persons start at the same time and travel toward a place 90 miles distant. A traveled one mile per hour faster than B, and reached the place one hour before him. At what rate did each travel?

20. A person found that he had in his purse, in silver and copper coins, just one dollar. Each copper coin was worth as many cents as there were silver coins, and each silver coin was worth as many cents as there were copper coins. There were in all 27 coins. How many were there of each?

FORMATION OF QUADRATIC EQUATIONS.

288. 1. What are the factors of x^2+5x+6?

2. If $x^2+4x-5=0$, to what is each factor equal?

3. If $x-1=0$, and $x+5=0$, what are the values of x, or the roots of the equation?

4. If $x^2+4x=5$, what is the form when 5 is transposed to the first member?

5. How is the term that does not contain the unknown quantity formed from the roots? How is the coëfficient of the first power of the unknown quantity formed from the roots?

289. When the unknown quantities are collected in the first member, and the known quantities united in the second member, the term of the second member is called the **Absolute Term.**

290. By the solution of the general equation $x^2+bx=c$, the facts developed may be shown in general:

$$x^2+bx=c$$

$$x=-\frac{b}{2}+\sqrt{c+\frac{b^2}{4}}$$

$$x=-\frac{b}{2}-\sqrt{c+\frac{b^2}{4}}$$

The sum of the two roots gives $-b$. Hence,

PRINCIPLES.—1. *The sum of the two roots of an affected quadratic is equal to the coëfficient of the first power of the unknown quantity with the sign changed.*

2. *The product of the two roots is equal to the absolute term with the sign changed.*

EXAMPLES.

1. Form a quadratic equation whose roots are 2 and 3.

PROCESS.

$3 + 2 = 5$

$3 \times 2 = 6$

$x^2 - 5x = -6$

EXPLANATION.—Since the coëfficient of the first power is the sum of the roots, with the sign changed, and the absolute term is the product of the roots with the sign changed, $x^2 - 5x = -6$ is a quadratic fulfilling the required conditions.

Form quadratic equations whose roots are as follows:

2. 3 and 4.	8. -2 and 6.
3. 2 and -5.	9. -3 and -7.
4. 3 and 7.	10. a and $-b$.
5. -4 and -6.	11. b and $-c$.
6. -3 and 2.	12. $\sqrt{5}$ and $2\sqrt{5}$.
7. -4 and -5.	13. $2 + \sqrt{7}$ and $2 - \sqrt{7}$.

SIMULTANEOUS QUADRATIC EQUATIONS.

291. A **Homogeneous Equation** is an equation in which the sum of the exponents of the unknown quantities in each term which contains unknown quantities, is the same.

Thus, $x^2 + 2y^2$ and $xy + y^2$ are homogeneous equations.

292. Simultaneous Quadratic Equations can usually be solved by the rules for quadratics, if they belong to one of the following classes:

1. *When one is simple and the other quadratic.*
2. *When the unknown quantities in each equation are combined in a similar manner.*
3. *When each equation is homogeneous and quadratic.*

293. The following solutions will illustrate the processes in many of the ordinary forms of simultaneous quadratics:

(I.) Simple and Quadratic.

1. Given $\left\{\begin{array}{r} x + y = 5 \\ 2x^2 + y^2 = 17 \end{array}\right\}$, to find the values of x and y.

SOLUTION.

$$x + y = 5 \qquad (1)$$
$$2x^2 + y^2 = 17 \qquad (2)$$

From (1),	$x = 5 - y$	(3)
	$2x^2 = 50 - 20y + 2y^2$	(4)
Substituting in (2),	$50 - 20y + y^2 + 2y^2 = 17$	(5)
Collecting terms, etc.,	$3y^2 - 20y = -33$	(6)
Solving,	$y = 3 \text{ or } 3\frac{2}{3}$	(7)
Substituting in (1),	$x = 2 \text{ or } 1\frac{1}{3}$	(8)

(II.) Unknown quantities similarly combined.

2. Given $\left\{\begin{array}{r} x + y = 5 \\ xy = 6 \end{array}\right\}$, to find the values of x and y.

SOLUTION.

$$x + y = 5 \qquad (1)$$
$$xy = 6 \qquad (2)$$

Squaring (1),	$x^2 + 2xy + y^2 = 25$	(3)
Multiplying (2) by 4,	$4xy = 24$	(4)
Subtracting (4) from (3),	$x^2 - 2xy + y^2 = 1$	(5)
Extracting square root of (5),	$x - y = \pm 1$	(6)
From (1),	$x + y = 5$	(7)
Adding (7) and (6),	$2x = 6 \text{ or } 4$	(8)
	$x = 3 \text{ or } 2$	(9)
Subtracting (6) from (7),	$2y = 4 \text{ or } 6$	(10)
	$y = 2 \text{ or } 3$	(11)

3. Given $\left\{\begin{array}{r} x + \quad y = 5 \\ x^2y + xy^2 = 30 \end{array}\right\}$, to find the values of x and y.

SOLUTION.

$$x + y = 5 \qquad (1)$$
$$x^2y + xy^2 = 30 \qquad (2)$$

Factoring (2), $\quad xy(x + y) = 30 \qquad (3)$

Dividing (3) by (1), $\quad xy = 6 \qquad (4)$

Squaring (1), $\quad x^2 + 2xy + y^2 = 25 \qquad (5)$

Multiplying (4) by 4, $\quad 4xy = 24 \qquad (6)$

Subtracting (6) from (5), $\quad x^2 - 2xy + y^2 = 1 \qquad (7)$

Extracting square root of (7), $\quad x - y = \mp 1 \qquad (8)$

Adding (8) and (1), $\quad 2x = 6 \text{ or } 4 \qquad (9)$

$\quad x = 3 \text{ or } 2 \qquad (10)$

Subtracting (8) from (1), $\quad 2y = 4 \text{ or } 6 \qquad (11)$

$\quad y = 2 \text{ or } 3 \qquad (12)$

4. Given $\left\{\begin{array}{l} x \ + y \ = 8 \\ x^3 + y^3 = 152 \end{array}\right\}$, to find the values of x and y.

SOLUTION.

$$x + y = 8 \qquad (1)$$
$$x^3 + y^3 = 152 \qquad (2)$$

Dividing (2) by (1), $\quad x^2 - xy + y^2 = 19 \qquad (3)$

Squaring (1), $\quad x^2 + 2xy + y^2 = 64 \qquad (4)$

Subtracting (3) from (4), $\quad 3xy = 45 \qquad (5)$

$\quad xy = 15 \qquad (6)$

Subtracting (6) from (3), $\quad x^2 - 2xy + y^2 = 4 \qquad (7)$

Extracting square root of (7) $\quad x - y = \pm 2 \qquad (8)$

Adding (8) and (1), $\quad 2x = 10 \text{ or } 6 \qquad (9)$

$\quad x = 5 \text{ or } 3 \qquad (10)$

Subtracting (8) from (1), $\quad 2y = 6 \text{ or } 10 \qquad (11)$

$\quad y = 3 \text{ or } 5 \qquad (12)$

(III.) Homogeneous equations.

5. Given $\left\{\begin{array}{l} x^2 - xy = 15 \\ 2xy - y^2 = 16 \end{array}\right\}$, to find the values of x and y.

SOLUTION.

$$x^2 - xy = 15 \qquad (1)$$
$$2xy - y^2 = 16 \qquad (2)$$

Assume, $x = vy \qquad (3)$

Substituting vy in (1), $v^2y^2 - vy^2 = 15 \qquad (4)$

Substituting vy in (2), $2vy^2 - y^2 = 16 \qquad (5)$

From (4), $y^2 = \dfrac{15}{v^2 - v} \qquad (6)$

From (5), $y^2 = \dfrac{16}{2v - 1} \qquad (7)$

Equating (6) and (7), $\dfrac{16}{2v-1} = \dfrac{15}{v^2 - v} \qquad (8)$

Clearing of fractions and reducing, $16v^2 - 46v = -15 \qquad (9)$

Whence, $v = \dfrac{5}{2}$ or $\dfrac{3}{8} \qquad (10)$

Substituting the value of v in (7), $y^2 = \dfrac{16}{5-1}$ or $4 \qquad (11)$

And $y^2 = \dfrac{16}{\frac{3}{4} - 1}$ or $-64 \qquad (12)$

$$y = \pm 2 \text{ or } \pm 8\sqrt{-1}$$
$$x = 5 \text{ or } \pm 20\sqrt{-1}$$

Find the values of the unknown quantities in the following:

6. $\left\{\begin{array}{l} x - y = 4. \\ x^2 - y^2 = 32. \end{array}\right\}$

7. $\left\{\begin{array}{l} x + y = 5. \\ x^2 + y^2 = 13. \end{array}\right\}$

8. $\begin{cases} xy = 15. \\ 2x + y = 13. \end{cases}$

9. $\begin{cases} xy = 6. \\ 2x - 3y = 9. \end{cases}$

10. $\begin{cases} x + 2y = 7. \\ 2x^2 - y^2 = 14. \end{cases}$

11. $\begin{cases} 2x + 3y = 22. \\ 2xy = 40. \end{cases}$

12. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{3}{10}. \\ \dfrac{1}{x^2} + \dfrac{1}{y^2} = \dfrac{1}{20}. \end{cases}$

13. $\begin{cases} xy = 24. \\ x^2 - 2y^2 = 4. \end{cases}$

14. $\begin{cases} xy = 10. \\ x - y = 3. \end{cases}$

15. $\begin{cases} xy = 12. \\ x + y = 7. \end{cases}$

16. $\begin{cases} x - y = 1. \\ x^3 - y^3 = 37. \end{cases}$

17. $\begin{cases} x + y = 4. \\ x^3 + y^3 = 28. \end{cases}$

18. $\begin{cases} x^3 + y^3 = 28. \\ x^2y + xy^2 = 12. \end{cases}$

19. $\begin{cases} x^3 - y^3 = 26. \\ xy^2 - x^2y = -6. \end{cases}$

20. $\begin{cases} x^2 - xy = 6. \\ x^2 + y^2 = 61. \end{cases}$

21. $\begin{cases} 3x^2 + xy = 18. \\ 4y^2 + 3xy = 54. \end{cases}$

22. $\begin{cases} x^2 + xy = 70. \\ xy - y^2 = 12. \end{cases}$

23. $\begin{cases} 2x + y = 22. \\ xy + 2y^2 = 120. \end{cases}$

24. $\begin{cases} x = 2y^2. \\ x - y = 15. \end{cases}$

25. $\begin{cases} x + 4y = 14. \\ y^2 - 2y + 4x = 11. \end{cases}$

26. $\begin{cases} 4xy + x^2y^2 = 96. \\ x + y = 6. \end{cases}$

27. $\begin{cases} x^2 + y^2 = 52. \\ x + y + xy = 34. \end{cases}$

28. $\begin{cases} x^2 + y^2 = 13. \\ xy + y^2 = 15. \end{cases}$

29. $\begin{cases} \sqrt[3]{x} + \sqrt[3]{y} = 6. \\ x + y = 72. \end{cases}$

30. $\begin{cases} x^3 - y^3 = 56. \\ x - y = \dfrac{16}{xy}. \end{cases}$

31. $\begin{cases} x + y = 3. \\ x^4 + y^4 = 17. \end{cases}$

32. Given $\left\{\begin{array}{r}(x-y)(x^2+y^2)=13 \\ x^2y-\cdot xy^2=6\end{array}\right\}$, to find x and y.

33. Given $\left\{\begin{array}{r}x^2+x+y=18-y^2 \\ xy=6.\end{array}\right\}$, to find x and y

34. Given $\left\{\begin{array}{l}x+\sqrt{xy}+y=19 \\ x^2+xy+y^2=133\end{array}\right\}$, to find x and y.

35. Given $\left\{\begin{array}{l}x^2+4y^2=256-4xy \\ 3y^2-x^2=39.\end{array}\right\}$, to find x and y.

36. Given $\left\{\begin{array}{r}x+y+\sqrt{x+y}=6 \\ x^2+y^2=10\end{array}\right\}$, to find x and y.

37. Given $\left\{\begin{array}{r}x^2+y^2-(x+y)=78 \\ xy+(x+y)=39\end{array}\right\}$, to find x and y.

38. Given $\left\{\begin{array}{r}x^2+y^2+4\sqrt{x^2+y^2}=45 \\ x^4+y^4=337\end{array}\right\}$, to find x and y.

PROBLEMS.

294. 1. The sum of two numbers is 8, and their product 12. What are the numbers?

2. The sum of two numbers is 12, and the sum of their squares 104. What are the numbers?

3. Divide 13 into two such parts that the sum of their square roots is 5.

4. The product of two numbers is 99, and their sum 20. What are the numbers?

5. The sum of two numbers is 100, and the difference of their square roots is 2. What are the numbers?

6. The difference of two numbers is 2, and the difference of their cubes is 56. What are the numbers?

7. Find two numbers whose sum multiplied by the second is 84, and whose difference multiplied by the first is 16.

8. The product of two numbers is 48, and the difference of their cubes is 37 times the cube of their difference. What are the numbers?

9. The sum of two numbers is a, and the sum of their squares is b. What are the numbers?

10. What two numbers are there such that their sum increased by their product is 34, and the sum of their squares diminished by their sum is 42?

11. There is a number expressed by two digits, such that the sum of the squares of the digits is equal to the number increased by the product of its digits, and if 36 is added to the number the digits will be reversed. What is the number?

12. From two places, distant 720 miles, A and B set out to meet each other. A traveled 12 miles a day more than B, and the number of days before they met was equal to one-half the number of miles B went per day. How many miles did each travel per day?

13. A merchant received $12 for a quantity of linen, and an equal sum, at 50 cents a yard less, for a quantity of cotton. The cotton exceeded the linen by 32 yards. How many yards did he sell of each?

14. A farmer has a field 18 rods long and 12 rods wide, which he wishes to enlarge so that it may contain twice its former area by making a uniform addition on all sides. What will be the sides of the field when it is enlarged?

15. A merchant bought a piece of cloth for $147, from which he cut off 12 yards which were damaged, and sold the remainder for $120.25, gaining 25 cents on each yard sold. How many yards did he buy? How much did it cost per yard?

16. The fore wheel of a carriage makes 6 revolutions

more than the hind wheel, in going 360 feet. If the circumference of each wheel had been 3 feet greater, the fore wheel would have made only 4 revolutions more than the hind wheel in going that distance. What is the circumference of each wheel?

17. Find two numbers such that their sum, product, and difference of their squares shall all be equal.

18. The joint capital of A and B was $416. A's money was in trade 9 months, and B's 6 months. When they shared stock and gain, A received $228 and B $252. What was the capital of each?

19. A rectangular piece of ground has a perimeter of 100 rods, and its area is 589 square rods. What are its length and breadth?

20. Twenty persons sent together $48 to a benevolent society. One-half the amount was contributed by women, and the other half by men; but each man gave a dollar more than each woman. How many women contributed? How many men? What was the contribution of each?

21. In a purse containing 9 coins, some are of gold, others of silver. Each gold coin is worth as many dollars as there are silver coins, and each silver coin is worth as many cents as there are gold coins, and the value of the whole is $20.20. How many are there of each?

22. The difference of two numbers is 15, and half their product equals the cube of the smaller. What are the numbers?

23. A and B set out from two places, C and D, at the same time. A started from C and traveled through D in the same direction in which B traveled. When A overtook B, it was found that they had together traveled 60 miles, that A had passed through D 5 hours before, and that it would have required 20 hours for B to return to C at the rate he had been traveling. What was the distance from C to D?

RATIO.

295. 1. How does \$2 compare with \$6? 3 pounds with 9 pounds? 4 tons with 8 tons?

2. How does $3a$ compare with $6a$? $5a$ with $15a$? $4x$ with $12x$? $5ax$ with $10a^2x$? $3ax^2$ with $6a^2x^2$?

3. What relation has $2a$ to $4a$? $3a$ to $6a$? $3a^2$ to $15a^2$?

4. What is the relation of $8x$ to $16x$? $5y$ to $10y$? $4axy$ to $8a^2x^2y^2$?

5. How does $8a$ compare with $2a$? What is the relation of $8a$ to $2a$?

6. How does $9xy$ compare with $3xy$? What is the relation of $9xy$ to $3xy$?

7. What is the relation of $2a$ to $4a$? What is the relation *between* $2a$ and $4a$?

8. What is the relation of $3x^2$ to $9x^2$? What is the relation *between* $3x^2$ and $9x^2$?

9. What is the relation of $(x+y)$ to $2(x+y)$? Between $(x+y)$ and $2(x+y)$?

10. What is the relation of $(a+x)$ to $b(a+x)$? Between $(a+x)$ and $b(a+x)$?

11. What is the relation of $3x^2y$ to $9x^3y$? Between $3x^2y$ and $9x^3y$?

DEFINITIONS.

296. Ratio is the relation of one quantity to another of the same kind.

1. This relation is expressed either as the *quotient* of one quantity

divided by the other, and is called *Geometrical Ratio* or simply *Ratio*, or as the *difference* between two quantities, and is called *Arithmetical Ratio.*

2. When it is required to determine *what the relation of one quantity to another is*, it is evident that the *first* is the *dividend* and the *second* the *divisor*. Thus, when the question is, "What is the relation of $5a$ to $10a$?" the answer is $\frac{1}{2}$.

3. When it is required to determine the relation *between* two quantities, *either* may be regarded as *dividend* or *divisor*. Thus, when the question is, "What is the relation *between* $5a$ and $10a$?" the answer is $\frac{1}{2}$, or 2.

4. The *first* quantity is commonly regarded as the dividend, although whether it should be such or not depends upon the question asked, as shown in Notes 2 and 3.

297. The **Terms of a Ratio** are the quantities compared.

298. The **Sign** of ratio is a colon (:).

Thus, the ratio between $12a$ and $6a$ is expressed $12a : 6a$.

The colon is sometimes regarded as derived from the sign of division, by omitting the line.

299. The **Antecedent** is the first term of the ratio.

Thus, in the expression $5a : 3a$, $5a$ is the antecedent.

300. The **Consequent** is the second term of the ratio.

Thus, in the expression $5a : 3a$, $3a$ is the consequent.

301. A **Couplet** is the antecedent and consequent taken together.

302. A **Simple Ratio** is the ratio of two quantities.

Thus, $(2a + b) : 3x$, and $3x : 4y$, are simple ratios.

303. Ratios are **compounded** by multiplying the antecedents of the ratios together, for the antecedent of the

new ratio, and the consequents for the consequent of the new ratio.

Thus, if the ratios $a : c$ and $a : b$ are compounded, the resulting ratio is $a^2 : bc$.

304. The ratio of the squares of two quantities is called the **Duplicate** ratio of the quantities; the ratio of their cubes, their **Triplicate** ratio.

Thus, $a^2 : b^2$ and $a^3 : b^3$ are respectively the duplicate and triplicate ratios of a and b.

305. Since the ratio of two quantities, as the ratio of a to b, may be expressed by a fraction, as $\frac{a}{b}$, it follows that the changes which may be made upon a fraction without altering its value, may be made upon the terms of a ratio without changing the ratio of the terms, since the numerator is the antecedent and the denominator the consequent. Hence—

PRINCIPLE.—*Multiplying or dividing both terms of a ratio by the same quantity does not change the ratio of the terms.*

EXAMPLES.

1. What is the ratio of $3a$ to $6a$? $5a$ to $10a$?
2. What is the ratio of $7x$ to $35x$? $12ay$ to $13a$?
3. If the antecedent is $15a$, and the consequent $20a$, what is the ratio?
4. What is the ratio of $\frac{1}{2}$ to $\frac{1}{4}$? $\frac{1}{4}$ to $\frac{1}{3}$? $\frac{5}{6}$ to $\frac{2}{3}$?

When fractions are reduced to similar fractions they have the ratio of their numerators.

5. When the antecedent is $2a$, and the ratio is $\frac{1}{2}$, what is the consequent?

PROPORTION.

306. 1. What two numbers have the same relation to each other as 3 to 6? As 2 to 8? As 5 to 15? As 8 to 24?

2. What two quantities have the same relation to each other as $2a$ to $4a$? As $3b$ to $6b$? As $8b$ to $16b$?

3. What quantity has the same relation to $6a$ that $2b$ has to $4b$?

4. What quantity has the same relation to $10x$ that $3y$ has to $9y$?

5. What quantity has the same relation to $4x^2$ that $5a$ has to $10a$?

6. What quantity has the same relation to $5a$ that $5b$ has to $5ab$?

7. What quantity has the same relation to $4ax$ that $3x$ has to $6xy$?

8. What two quantities have the same ratio to each other that $5ay$ has to $10ay^2$?

9. What two quantities have the same ratio to each other that $8ax$ has to $4ax^2$?

10. How have the two ratios in each of the several examples given above compared in value?

DEFINITIONS.

307. A **Proportion** is an equality of ratios.

Thus, $5 : 6 = 10 : 12$, and $5xy : 10xy = 4az : 8az$, are proportions.

308. The **Sign** of proportion is a double colon (::).

This sign has been supposed to be the extremities of the lines which form the sign of equality. It is written between the ratios thus: $x : y :: 2a : 2b$.

The sign of equality is frequently used instead of the double colon.

309. The **Antecedents** of a proportion are the antecedents of the ratios which form the proportion.

Thus, in the proportion $a : b :: c : d$, a and c are the *antecedents*.

310. The **Consequents** of a proportion are the consequents of the ratios which form the proportion.

Thus, in the proportion $a : b :: c : d$, b and d are the *consequents*.

311. The **Extremes** of a proportion are the first and fourth terms of the proportion.

Thus, in the proportion $a : b :: c : d$, a and d are the *extremes*.

312. The **Means** of a proportion are the second and third terms of the proportion.

Thus, in the proportion $a : b :: c : d$, b and c are the *means*.

313. A **Mean Proportional** is a quantity which serves as both means of a proportion.

Thus, in the proportion $a : b :: b : c$, b is a *mean proportional*.

314. Since a proportion is an equality of ratios, and the ratio of two quantities is found by dividing the antecedent by the consequent, it follows that—

PRINCIPLE.—*A proportion may be expressed as an equation in which both members are fractions.*

Thus, the proportion $a : b :: c : d$ may be expressed as $\frac{a}{b} = \frac{c}{d}$.

315. Since a proportion may be regarded as an equation in which both members are fractions, it follows that—

PRINCIPLE.—*The changes that may be made upon a proportion without destroying the proportion, are based upon the changes that may be made upon an equation without destroying the equality, and upon a fraction without altering its value.*

PRINCIPLES OF PROPORTION.

316. 1. Let any four quantities form a proportion; as, $a : b :: c : d$.

2. In what other manner may this proportion be expressed? See Art. 314. Express it in that manner.

3. Clear the equation of fractions.

4. What does each member of the resulting equation contain?

5. How are the members of the equation produced from the terms of the proportion?

PRINCIPLE 1.—*In any proportion the product of the extremes is equal to the product of the means.*

Thus, when $a : b :: c : d$, $ad = bc$.

Since a *mean* proportional serves as both means of a proportion, as $a : b :: b : c$, it follows that

The product of the extremes is equal to the square of the mean proportional.

DEMONSTRATION OF PRINCIPLE I.

Let $a : b :: c : d$ represent any proportion.

Then, $\frac{a}{b} = \frac{c}{d}$.

Clearing of fractions, $ad = bc$. Therefore, etc.

NUMERICAL ILLUSTRATION.

$$3 : 6 :: 8 : 16$$
$$3 \times 16 = 8 \times 6$$
$$48 = 48$$

---

317. 1. Change the proportion $a : b :: c : d$ into an equation, according to Principle 1.

2. Since, then, $ad = bc$, how may the value of a be found? How the value of d? What are a and d of the proportion?

3. How, then, may either extreme of a proportion be found? How may either mean be found?

PRINCIPLE 2.—*Either extreme is equal to the product of the means divided by the other extreme. Either mean is equal to the product of the extremes divided by the other mean.*

Thus, when $a : b :: c : d$, $a = \frac{bc}{d}$, $d = \frac{bc}{a}$, $b = \frac{ad}{c}$, $c = \frac{ad}{b}$.

Demonstrate Prin. 2, and illustrate its truth with numbers.

---

318. 1. If $ad = bc$, what will be the resulting equation when both members are divided by bd?

2. Express the resulting equation as a proportion.

3. What does ad, the first member of the equation, form in the proportion? What bc?

PRINCIPLE 3.—*If the product of two quantities is equal to the product of two other quantities, two of them may be made the extremes, and the other two the means, of a proportion.*

Thus, when $ad = bc$, $a : b :: c : d$.

Demonstrate Prin. 3, and illustrate its truth with numbers.

319. 1. Change the general proportion $a:b::c:d$ into an equation, according to Principle 1.

2. Divide the members of the equation by cd.

3. Express the result as a proportion.

4. What change has taken place in the order of antecedents and consequents, compared with the original proportion?

PRINCIPLE 4.—*If four quantities are in proportion, the antecedents will have the same ratio to each other as the consequents.*

Thus, when $a:b::c:d$, $a:c::b:d$.

When the antecedents have the same ratio as the consequents, the quantities are said to be in proportion by *Alternation.*

Demonstrate Prin. 4, and illustrate its truth with numbers.

---

320. 1. Change the general proportion $a:b::c:d$ into an equation, according to Principle 1.

2. Divide the members of the equation by ac.

3. Express the result as a proportion.

4. What change has taken place in the order of the terms in each couplet, compared with the original proportion?

PRINCIPLE 5.—*If four quantities are in proportion, the second will be to the first as the fourth to the third.*

Thus, when $a:b::c:d$, $b:a::d:c$.

When the second is to the first as the fourth is to the third, or when the terms of each ratio are written in the *inverse* order, the quantities are said to be in proportion by *Inversion.*

Demonstrate Prin. 5, and illustrate its truth with numbers.

321. 1. Express the proportion $a:b::c:d$ as a fractional equation.

2. Add 1 to each member of the equation.

3. Reduce each of the mixed quantities to the fractional form.

4. Express the result as a proportion.

5. How are the terms of this proportion formed from the terms of the original proportion?

6. Since, when $a:b::c:d$, $b:a::d:c$ (Prin. 5), if the changes just indicated are made in the second proportion, how may the terms of the resulting proportion be obtained from the terms of the original proportion?

PRINCIPLE 6.—*If four quantities are in proportion, the sum of the terms of the first ratio is to either term of the first ratio as the sum of the terms of the second ratio is to the corresponding term of the second ratio.*

Thus, when $a:b::c:d$, $a+b:b::c+d:d$ and $a+b:a::c+d:c$.

When the sum of the terms of a ratio is to one of the terms as the sum of the terms of another ratio is to its corresponding term, the quantities are said to be in proportion by *Composition.*

Demonstrate Prin. 6, and illustrate its truth with numbers.

---

322. 1. Express the proportion $a:b::c:d$ as a fractional equation.

2. Subtract 1 from each member of the equation.

3. Reduce each of the mixed quantities to the fractional form.

4. Express the result as a proportion.

5. How are the terms of this proportion formed from the terms of the original proportion?

6. Since, when $a:b::c:d$, $b:a::d:c$ (Prin. 5), if

the changes just indicated are made in the second proportion, how may the terms of the resulting proportion be obtained from the terms of the original proportion?

PRINCIPLE 7.—*If four quantities are in proportion, the difference between the terms of the first ratio is to either term of the first ratio as the difference between the terms of the second ratio is to the corresponding term of the second ratio.*

Thus, when $a:b::c:d$, $a-b:b::c-d:d$ and $a-b:a::c-d:c$.

When the difference of the terms of a ratio is to one of the terms as the difference of the terms of another ratio is to its corresponding term, the quantities are said to be in proportion by *Division.*

Demonstrate Principle 7, and illustrate by numerical examples.

---

323. 1. Change the proportion $a:b::c:d$, according to Principle 6. Express the resulting proportion.

2. Change the same proportion according to Principle 7. Express the resulting proportion.

3. Change these proportions to fractional equations.

4. Divide the first equation by the second.

5. Express the result as a proportion.

6. How are the terms of this proportion formed from the terms of the original proportion?

PRINCIPLE 8.—*If four quantities are in proportion, the sum of the quantities which form the first couplet is to their difference as the sum of the quantities which form the second couplet is to their difference.*

Thus, when $a:b::c:d$, $a+b:a-b::c+d:c-d$.

Demonstrate Principle 8, and illustrate by numerical examples.

324. 1. Express the proportion $a:b::c:d$ as a fractional equation.

2. Raise both members to the nth power.

3. Express the nth root of both members.

4. Express each of the equations as a proportion.

5. How may these proportions be formed from the original proportion?

PRINCIPLE 9.—*If four quantities are in proportion, the same powers of those quantities, or the same roots, will be in proportion.*

Thus, when $a:b::c:d$, $a^n : b^n :: c^n : d^n$, and $a^{\frac{1}{n}} : b^{\frac{1}{n}} :: c^{\frac{1}{n}} : d^{\frac{1}{n}}$.

Demonstrate Principle 9, and illustrate by numerical examples.

---

325. 1. Express the proportion $a:b::c:d$ as a fractional equation.

2. What may be done to a fraction without changing its value?

3. Multiply the terms of the first fraction by m, and the terms of the second by n.

4. Express the result as a proportion.

5. How are the terms of this proportion formed from the original proportion?

PRINCIPLE 10.—*If four quantities are in proportion, any equi-multiple of the terms of the first couplet will be proportional to any equi-multiple of the terms of the second couplet.*

Thus, when $a:b::c:d$, $ma:mb::nc:nd$.

Demonstrate Principle 10, and illustrate by numerical examples.

326. 1. Express the proportions $a:b::c:d$ and $x:y::z:w$ as fractional equations.

2. Multiply the resulting equations together.

3. Express the resulting equation as a proportion.

4. How are the terms of this proportion formed from the terms of the original proportions $a:b::c:d$ and $x:y::z:w$?

PRINCIPLE 11.—*If four quantities in proportion are multiplied term by term by four other quantities in proportion, the products will be in proportion.*

Thus, when $a:b::c:d$ and $x:y::z:w$, $ax:by::cz:dw$.

Demonstrate Principle 11, and illustrate by numerical examples.

Prove that the quotients will be in proportion if the proportions are *divided* term by term.

---

327. 1. Express the proportions $a:b::c:d$ and $a:b::e:f$ as fractional equations.

2. Since the first members of the equations are equal, what will the second members form?

3. Express the resulting equation as a proportion.

4. How are the terms of this proportion formed from the terms of the original proportions $a:b::c:d$ and $a:b::e:f$?

PRINCIPLE 12.—*If two proportions have a couplet in each the same, the other couplets will form a proportion.*

Thus, when $a:b::c:d$, and $a:b::e:f$, then $c:d::e:f$.

Demonstrate Principle 12, and illustrate by numerical examples.

EXAMPLES.

328. 1. In $5 : 8 :: 4 : x$, find the value of x.

SOLUTION.

$$5 : 8 :: 4 : x$$

By Prin. 1, $5x = 32$

Therefore, $x = 6\frac{2}{5}$

In solving examples like the following, the student should employ as many of the Principles of Proportion as are applicable.

2. In $3 : x :: 4 : 6$, find the value of x.

3. In $x : 5 :: 3 : 10$, find the value of x.

4. In $4 : 6 :: x : 4$, find the value of x.

5. In $3 : x :: x : 12$, find the value of x.

6. In $x : 4 :: x^2 : 6$, find the value of x.

7. In $x - 1 : x - 2 :: 2x + 1 : x + 2$, find the value of x.

8. Divide \$40 between two men so that their shares shall be in the proportion of 3 to 7.

9. There are two numbers in the ratio of 2 to 3, and if 3 is added to each, the ratio of the resulting numbers will be 5 to 7. What are the numbers?

10. There are two numbers which have to each other the ratio of 3 to 5; and if 4 is added to each, the results will have the ratio of 2 to 3. What are the numbers?

11. Mr. A's crop of wheat was to his crop of oats as 2 to 3. If he had raised 50 bushels more of each, the quantity of wheat would have been to the quantity of oats as 5 to 7. How many bushels of each kind of grain did he raise?

12. Find two numbers such that the greater is to the less as their sum is to 6, and the greater is to the less as their difference is to 2.

SOLUTION.

Let x = the greater; y = the less.

By the conditions, $x : y :: x + y : 6$ (1)

$x : y :: x - y : 2$ (2)

By Prin. 12, $x + y : 6 :: x - y : 2$ (3)

By Prin. 4, $x + y : x - y :: 6 : 2$ (4)

By Prin. 8, $2x : 2y :: 8 : 4$ (5)

By Prin., Art. 305, $x : y :: 2 : 1$ (6)

From (1) and (6), Prin. 12, $x + y : 6 :: 2 : 1$ (7)

From (2) and (6), Prin. 12, $x - y : 2 :: 2 : 1$ (8)

From (7), $x + y = 12$ (9)

From (8), $x - y = 4$ (10)

Whence, $2x = 16$ (11), $x = 8$ (12), $y = 4$ (13)

13. The product of two numbers is 20, and the difference of their squares is to the square of their difference as $9 : 1$.

SOLUTION.

Let x = the greater; y = the less.

By the conditions, $\begin{cases} xy = 20 & (1) \\ x^2 - y^2 : (x - y)^2 :: 9 : 1 & (2) \end{cases}$

Dividing first couplet by

$(x - y)$, Prin., Art. 305, $x + y : x - y :: 9 : 1$ (3)

By Prin. 8, $2x : 2y :: 10 : 8$ (4)

By Prin., Art. 305, $x : y :: 5 : 4$ (5)

By Prin. 1, $4x = 5y$ (6)

$x = \frac{5y}{4}$ (7)

Substituting in (1) $\frac{5y^2}{4} = 20$ (8)

$y^2 = 16$ (9)

$y = \pm 4$ (10)

$x = \pm 5$ (11)

14. Find two numbers such that their sum is 8 and their product is to the sum of their squares as 15 to 34.

15. Find two numbers whose difference is 3, and whose product is to the sum of their squares as 10 is to 29.

16. What two numbers are those whose sum is to their difference as 7 to 1, and whose product is to the sum as 24 to 7?

17. The sum of two numbers is 12, and their product is to the sum of their squares as 2 to 5. What are the numbers?

18. The sum of two numbers is 6, and the sum of their squares is to the square of their sum as 5 to 9. What are the numbers?

19. What two numbers are those whose product is 12, and the difference of whose cubes is to the cube of their difference as 37 to 1?

FRACTIONAL EQUATIONS SOLVED BY THE PRINCIPLES OF PROPORTION.

329. Since a proportion is an equality of ratios, and the ratio of two quantities may be expressed as a fraction, it is evident that the Principles of Proportion are applicable to equations which have both members fractions.

Regarding the numerator of each fraction as an antecedent, and the denominator as a consequent, the terms of each fraction as a couplet, and the equation as a proportion, the Principles of Proportion may be easily applied.

1. Given $\dfrac{x-\sqrt{x+1}}{x+\sqrt{x+1}}=\dfrac{5}{11}$, to find x.

SOLUTION.

$$\frac{x-\sqrt{x+1}}{x+\sqrt{x+1}}=\frac{5}{11} \qquad (1)$$

By Principle 8, the sum of the numerator and denominator of each member, divided by their difference, will form an equation.

Hence, $$\frac{2x}{2\sqrt{x+1}}=\frac{16}{6} \qquad (2)$$

By Prin., Art. 305, $$\frac{x}{\sqrt{x+1}}=\frac{8}{3} \qquad (3)$$

Squaring, $$\frac{x^2}{x+1}=\frac{64}{9} \qquad (4)$$

Clearing of fractions, etc., $$9x^2-64x=64 \qquad (5)$$

Whence $$x=8 \text{ or } -\frac{8}{9}$$

2. Given $\dfrac{\sqrt{x+a}-\sqrt{x-a}}{\sqrt{x+a}+\sqrt{x-a}}=\dfrac{x}{2a}$, to find x.

SOLUTION.

$$\frac{\sqrt{x+a}-\sqrt{x-a}}{\sqrt{x+a}+\sqrt{x-a}}=\frac{x}{2a} \qquad (1)$$

By Prin. 8, $$\frac{2\sqrt{x+a}}{2\sqrt{x-a}}=\frac{x+2a}{2a-x} \qquad (2)$$

By Prin., Art. 305, $$\frac{\sqrt{x+a}}{\sqrt{x-a}}=\frac{x+2a}{2a-x} \qquad (3)$$

Squaring, $$\frac{x+a}{x-a}=\frac{x^2+4ax+4a^2}{x^2-4ax+4a^2} \qquad (4)$$

By Prin. 8, $$\frac{2x}{2a}=\frac{2x^2+8a^2}{8ax} \qquad (5)$$

By Prin., Art. 305, $$\frac{x}{a}=\frac{x^2+4a^2}{4ax} \qquad (6)$$

Dividing denominators by a, $$x=\frac{x^2+4a^2}{4x} \qquad (7)$$

Whence $$x=\pm 2a\sqrt{\tfrac{1}{3}}$$

Solve the following by applying the Principles of Proportion when possible:

3. Given $\dfrac{\sqrt{6x}-2}{\sqrt{6x}+2}=\dfrac{4\sqrt{6x}-9}{4\sqrt{6x}+6}$, to find x.

4. Given $\dfrac{\sqrt{ax}-b}{\sqrt{ax}+b}=\dfrac{3\sqrt{ax}-2b}{3\sqrt{ax}+5b}$, to find x.

5. Given $\dfrac{\sqrt{4x+1}+\sqrt{4x}}{\sqrt{4x+1}-\sqrt{4x}}=9$, to find x.

In the solution the second member may be written as $\frac{9}{1}$.

6. Given $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}=b$, to find x.

7. Given $\dfrac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^2+1}+\sqrt{x^2-1}}=\dfrac{1}{2}$, to find x.

8. Given $\dfrac{3\sqrt{x}-4}{\sqrt{x}+2}=\dfrac{3\sqrt{x}+15}{\sqrt{x}+40}$, to find x.

Multiply the denominators by 3, and apply Prin. 7.

9. Given $\dfrac{\sqrt{x}+28}{\sqrt{x}+4}=\dfrac{\sqrt{x}+38}{\sqrt{x}+6}$, to find x.

10. Given $\dfrac{a+x+\sqrt{2ax+x^2}}{a+x-\sqrt{2ax+x^2}}=b$, to find x.

11. Given $\dfrac{\sqrt{a^2+x^2}+x}{\sqrt{a^2+x^2}-x}=\dfrac{b}{c}$, to find x.

12. Given $\dfrac{\sqrt{a}+\sqrt{a-x^2}}{\sqrt{a}-\sqrt{a-x^2}}=a$, to find x.

PROGRESSIONS.

330. 1. How does each of the numbers 2, 4, 6, 8, 10, 12, compare with the number that follows it?

2. How may each of the numbers 4, 6, 8, etc., be obtained from the one that precedes it?

3. Write five numbers in succession, beginning with 2 and increasing regularly by 3.

4. Write five quantities in succession, beginning with x and increasing regularly by $2x$.

5. Write a series of five quantities, beginning with a and increasing regularly by d.

6. How does each of the quantities $2x$, $4x$, $8x$, $16x$, compare with the one that precedes it?

7. Write a series of six quantities, beginning with $2a$ and increasing by a constant multiplier $3a$.

8. Write a series of six quantities, beginning with a and increasing by a constant multiplier r.

DEFINITIONS.

331. A **Series** of quantities is quantities in succession, each derived from the preceding according to some fixed law.

332. The first and last terms of a series are called the *extremes*, the intervening terms the *means*.

Thus, in the series a, $a+d$, $a+2d$, $a+3d$, the quantities a and $a+3d$ are *extremes*, and the others are *means*.

333. An **Ascending Series** is one in which the quantities *increase* regularly from the first term.

Thus, 2, 4, 8, 16, and a, $a+d$, $a+2d$, etc., are ascending series.

334. A **Descending Series** is one in which the quantities *decrease* regularly from the first term.

Thus, 24, 12, 6, 3, and a, $a-d$, $a-2d$, $a-3d$, are descending series.

ARITHMETICAL PROGRESSION.

335. An **Arithmetical Progression** is a series of quantities which increase or decrease by the addition or subtraction of a constant quantity.

Thus, 4, 6, 8, 10, 12, 14, and a, $a-d$, $a-2d$, $a-3d$, are arithmetical progressions.

336. The constant quantity which is added or subtracted is called the **Common Difference.**

Thus, in the progression 2, 4, 6, 8, 10, the common difference is 2.

CASE I.

337. To find the last term.

1. In the arithmetical progression 2, 4, 6, 8, 10, what is the common difference? How is the second term obtained from the first? How is the third term obtained from the first? How is the fourth?

2. In the arithmetical progression x, $x+2$, $x+4$, $x+6$, what is the common difference? How many times does the common difference enter into the second term? How many times into the third term? How many times into the fourth term?

3. In the series a, $a+d$, $a+2d$, $a+3d$, how is the second term formed from the first term? The third term? The fourth term?

4. Since, in an arithmetical series in which a is the first term and d the common difference, the first three terms are a, $a+d$, $a+2d$, what is the fourth term? The seventh term? The eleventh term? Any term?

5. If the above series were descending, the first three terms would be a, $a-d$, $a-2d$. What would be the fifth term? The seventh term? The eleventh term? Any term?

338. Let a represent the first term, d the common difference, l the last term, and n the number of terms. Then, since each term contains the first term, increased or diminished by the common difference multiplied by the number of terms less 1, according as the series is ascending or descending, the rule for finding the last term may be expressed by the following formula:

$$l = a \pm (n-1)d. \quad \text{That is,}$$

The last term is equal to the first term increased by the common difference multiplied by the number of terms less 1, when the series is ascending, or decreased by the common difference multiplied by the number of terms less 1, when the series is descending.

EXAMPLES.

1. Find the 15th term of the series 1, 3, 5, 7, etc.

PROCESS.

$l = a + (n-1)d$
$l = 1 + (15-1)2$
$l = 29$

EXPLANATION.—In this example $a = 1$, $d = 2$, and $n = 15$. Substituting these values in the formula, the value of l, or the last term, is 29.

2. Find the 18th term of the series 4, 7, 10, 13, etc.
3. Find the 12th term of the series 3, 7, 11, 15, etc.
4. Find the 10th term of the series $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3, etc.
5. Find the 13th term of the series $2\frac{1}{4}$, 3, $3\frac{3}{4}$, $4\frac{1}{2}$, etc.
6. Find the 12th term of the series 25, 23, 21, 19, etc.
7. Find the 20th term of the series 8, 4, 0, -4, etc.
8. Find the 30th term of the series a, $2a$, $3a$, $4a$, etc.
9. Find the 18th term of the series x, $3x$, $5x$, $7x$, etc.
10. Find the 15th term of the series $\frac{7}{8}$, $\frac{13}{16}$, $\frac{3}{4}$, $\frac{11}{16}$, etc.
11. Find the nth term of the series 1, 3, 5, 7, etc.

12. A boy agreed to work for 50 days, at 25 cents for the first day, and an increase of 3 cents per day. What were his wages the last day?

13. A body falls $16\frac{1}{2}$ feet the first second, 3 times as far the second second, 5 times as far the third second. How far will it fall the seventh second?

CASE II.

339. To find the sum of the terms.

1. What is the sum of the terms of the series 2, 4, 6, 8, 10? Since there are five terms, what is the average term? How does it compare with the sum of the first and last terms?

2. What is the sum of the terms of the series 3, 6, 9, 12, 15? Since there are five terms, what is the average term? How does it compare with the sum of the first and last terms?

3. What is the sum of the series 1, 3, 5, 7, 9, 11, 13? Since there are seven terms, what is the average term? How does it compare with the sum of the first and last terms?

4. How does the average term in any arithmetical series compare with the sum of the first and last terms?

340. The formula for the sum of an arithmetical series may be deduced as follows:

Let a represent the first term; d, the common difference; l, the last term; n, the number of terms; and s, the sum of the terms. Writing the sum of a series of four terms, we have

$$s = a + (a+d) + (a+2d) + (a+3d)$$

Inverting, $s = (a+3d) + (a+2d) + (a+d) + a$

Adding, $2s = (2a+3d) + (2a+3d) + (2a+3d) + (2a+3d)$

Whence, $2s = 4(2a+3d)$ or 4 times the sum of the first and last terms.

And in general, $2s = n(a+l)$

Whence, $s = \frac{n}{2}(a+l)$ or $n\left(\frac{a+l}{2}\right)$. That is,

The sum of any arithmetical progression is equal to one-half the sum of the extremes multiplied by the number of terms.

EXAMPLES.

1. What is the sum of the series 2, 4, 6, 8, etc., containing 12 terms?

PROCESS.

$$l = a + (n-1)d$$

$$l = 2 + (11 \times 2) = 24$$

$$S = n\left(\frac{a+l}{2}\right)$$

$$S = 12\left(\frac{2+24}{2}\right) = 156$$

EXPLANATION.—Since the last term is not given, it is found by the previous case to be 24. Then, by the formula given for obtaining the sum, it is found to be 156.

2. What is the sum of 12 terms of the series 1, 3, 5, 7, etc.?

3. What is the sum of 9 terms of the series 4, 6, 8, 10, etc.?

4. What is the sum of 8 terms of the series 5, 8, 11, 14, etc.?

5. What is the sum of 7 terms of the series 3, $4\frac{1}{2}$, 6, $7\frac{1}{2}$, etc.?

6. What is the sum of 8 terms of the series $3a$, $5a$, $7a$, $9a$, etc.?

7. What is the sum of 9 terms of the series $a+b$, $a+b+c$, $a+b+2c$, etc.?

8. What is the sum of n terms of the series x, $3x$, $5x$, $7x$, etc.?

9. What is the sum of 8 terms of the series 2, 1, 0, -1, -2, etc.?

10. A man walked 15 miles the first day, and increased his rate 3 miles per day. How far did he walk in 11 days?

11. How many strokes does a common clock strike in 12 hours?

12. A person received a gift of $100 per year from his birth until he was 21 years old. These sums were deposited in a bank, and drew simple interest at 6%. How much was due him when he became of age?

341. Formulas for finding any element.

By combining the fundamental formulas given in the previous cases, all problems which may arise in Arithmetical Progression may be solved.

When any three of the elements are given, the other two may be found.

Deduce the following formulas:

GIVEN.	REQUIRED.	FORMULAS.	
$a, d, n,$	$l, s.$	$l=a+(n-1)d.$	$s=\frac{1}{2}n(2a+(n-1)d).$
$l, d, n,$	$a, s.$	$a=l-(n-1)d.$	$s=\frac{1}{2}n(2l-(n-1)d).$
$a, n, l,$	$d, s.$	$d=\frac{l-a}{n-1}.$	$s=n\left(\frac{a+l}{2}\right).$
$d, n, s,$	$a, l.$	$a=\frac{2s-n(n-1)d}{2n}.$	$l=\frac{2s+n(n-1)d}{2n}.$
$a, n, s,$	$d, l.$	$d=\frac{2(s-an)}{n(n-1)}.$	$l=\frac{2s}{n}-a.$
$l, n, s,$	$d, a.$	$d=\frac{2(nl-s)}{n(n-1)}.$	$a=\frac{2s}{n}-l.$
$a, d, l,$	$n, s.$	$n=\frac{l-a}{d}+1.$	$s=\frac{(l+a)(l-a+d)}{2d}.$
$a, l, s,$	$n, d.$	$n=\frac{2s}{a+l}.$	$d=\frac{(l+a)(l-a)}{2s-(l+a)}.$
$a, d, s,$	$l, n.$	$l=-\frac{1}{2}d\pm\sqrt{2ds+(a-\frac{1}{2}d)^2}.$	$n=\frac{\pm\sqrt{(2a-d)^2+8ds}-2a+d}{2d}.$
$l, d, s,$	$a, n.$	$a=\frac{1}{2}d\pm\sqrt{(l+\frac{1}{2}d)^2-2ds}.$	$n=\frac{2l+d\pm\sqrt{(2l+d)^2-8ds}}{2d}.$

SPECIAL APPLICATIONS.

342. In solving some of the problems in Arithmetical Progression there are several ways of representing the unknown terms of the series.

1. When x represents the first term of a series, and y the common difference, the series is represented by

$$x,\ x+y,\ x+2y,\ x+3y, \text{ etc.}$$

2. When there are *three* terms in the series, the middle term may be represented by x, and the common difference by y; as, $x-y,\ x,\ x+y.$

3. When there are *five* terms in the series, the middle term may be represented by x, and the common difference by y; as, $x-2y,\ x-y,\ x,\ x+y,\ x+2y.$

4. When there are *four* terms in the series, $x-3y$ may represent the first term, and $2y$ the common difference; as,

$$x-3y,\ x-y,\ x+y,\ x+3y.$$

It is obvious that by this notation the *sum* of the quantities contains but one unknown quantity.

PROBLEMS.

1. There are three numbers in arithmetical progression, whose sum is 18 and the sum of whose squares is 116. What are the numbers?

SOLUTION.

Let $x-y=$ the first term.
$x=$ the second term.
$x+y=$ the third term.
$y=$ the common difference.

By the conditions, $\left\{\begin{array}{r} 3x=18 \quad (1) \\ 3x^2+2y^2=116 \quad (2) \end{array}\right.$

From (1), $x=6$ (3)

Substituting in (2), $108+2y^2=116$ (4)

Whence, $y=2$ (5)

Therefore, $\left\{\begin{array}{rl} x-y=4, & \text{1st term} \quad (6) \\ x=6, & \text{2d term} \quad (7) \\ x+y=8, & \text{3d term} \quad (8) \end{array}\right.$

2. The first term of an arithmetical series is 5, the last term 92, and the sum of the terms 1455. What is the number of terms?

3. The first term of an arithmetical series is 2, the last term 30, and the sum of the terms 160. What is the number of terms?

4. The first term of an arithmetical series is 16, the common difference $-3\frac{1}{2}$, and the sum of the terms 30. What is the number of terms?

5. The sum of three numbers in arithmetical progression is 15, and the product of the second and third is 35. What are the numbers?

6. The sum of three numbers in arithmetical progression is 9, and their product is 15. What are the numbers?

7. The sum of three numbers in arithmetical progression is 18, and the sum of their squares is 126. What are the numbers?

8. There are three numbers in arithmetical progression such that the product of the first and third is 16, and the sum of the squares of the numbers is 93. What are the numbers?

9. There are three numbers in arithmetical progression such that the first is 3, and the product of the first and third is 21. What are the numbers?

10. The sum of four numbers in arithmetical progression is 10, and their product is 24. What are the numbers?

11. There are four numbers in arithmetical progression such that the product of the first and fourth is 27, and the product of the second and third is 35. What are the numbers?

12. There are four numbers in arithmetical progression such that the product of the fourth number by the common

difference is 16, and the product of the second and third is 24. What are the numbers?

13. There are five numbers in arithmetical progression such that their sum is 40, and the sum of their squares 410. What are the numbers?

14. The sum of five numbers in arithmetical progression is 25, and their product is 945. What are the numbers?

15. The product of four numbers in arithmetical progression is 280, and the sum of their squares is 166. What are the numbers?

16. A number is expressed by three digits which are in arithmetical progression. If the number is divided by the sum of the digits, the quotient will be 26, and if 198 be added to the number, the digits will be inverted. What is the number?

GEOMETRICAL PROGRESSION.

343. A **Geometrical Progression** is a series of quantities which increase or decrease by a constant multiplier or divisor.

Thus, 2, 4, 8, 16, 32, and ab^3, ab^2, ab, a, are geometrical progressions.

344. The constant multiplier or divisor is called the **Ratio.**

Thus, in the progression 2, 4, 8, 16, 32, the ratio is 2.

CASE I.

345. To find the last term.

1. In the geometrical progression 2, 4, 8, 16, 32, what is the ratio? How is the second term obtained from the first?

How is the third term obtained from the first? How is the fourth term obtained from the first? How is the fifth term obtained from the first?

2. In the geometrical progression x, xy, xy^2, xy^3, xy^4, what is the ratio? How many times does the ratio enter as a factor into the second term? How many times into the third term? How many times into the fourth term? How many times into the fifth?

3. Since, in a geometrical series in which a is the first term and r the ratio, the first four terms are a, ar, ar^2, ar^3, what is the fifth term? The seventh term? The eleventh term? Any term?

346. Let a represent the first term, r the ratio, l the last term, and n the number of terms. Since each term contains the first term multiplied by the ratio used as a factor 1 less time than the number of terms, the rule for finding the last term may be expressed by the following formula:

$$l = ar^{n-1}.\quad \text{That is,}$$

The last term is equal to the first term, multiplied by the ratio raised to a power whose index is 1 less than the number of terms.

EXAMPLES.

1. Find the 8th term of the series 2, 4, 8, etc.

PROCESS.	
$l = ar^{n-1}$	EXPLANATION.—In this example $a = 2$, $r = 2$, and $n = 8$.
$l = 2 \times 2^7$	Substituting these values in the formula, the
$l = 256$	value of l, or the last term, is 256.

2. Find the 6th term of the series 5, 10, 20, etc.
3. Find the 9th term of the series 2, 4, 8, etc.
4. Find the 7th term of the series 3, 9, 27, etc.
5. Find the 10th term of the series 1, 2, 4, 8, etc.

6. Find the 7th term of the series $2a$, $4a^2$, $8a^3$, etc.
7. Find the 9th term of the series 3, $6ax$, $12a^2x^2$, etc.
8. Find the nth term of the series 1, 2, 4, 8, etc.
9. Find the nth term of the series 3, 12, 36, etc.
10. Find the 8th term of the series 3, 1, $\frac{1}{3}$, etc.
11. If a person should be hired for 8 days for $1 the first day, $3 for the next day, $9 for the third day, and so on, what would be his wages for the last day?
12. If a man begins business with a capital of $1000, and doubles it every three years, how much will he have at the end of 15 years?

CASE II.

347. To find the sum of a series.

1. In the geometrical series 5, 15, 45, 135, 405, what is the ratio?
2. If each term of this series is multiplied by the ratio, how will the terms of the product compare with the terms in the given series?
3. Since all the terms of both series except two are alike, if the sum of the terms of the given series is subtracted from the sum of the terms of the derived series, what terms will the remainder contain?
4. Since the sum of the given series was subtracted from the same series multiplied by the ratio, when the subtraction is performed, how many times the sum of the given series remains?
5. Since the sum, multiplied by the ratio — 1, is equal to the first term multiplied by the ratio raised to a power equal to the number of terms, and the product diminished by the first term, how may the sum of a geometrical series be found?

348. The formula for the sum of a geometrical series may be deduced as follows;

Let a represent the first term; r, the ratio; l, the last term; n, the number of terms; and s, the sum of the terms. Then,

$$s = a + ar + ar^2 + ar^3 \ldots\ldots + ar^{n-1} \qquad (1)$$

$$rs = \qquad ar + ar^2 + ar^3 \ldots\ldots + ar^{n-1} + ar^n \qquad (2)$$

Subtracting (1) from (2), $rs - s = ar^n - a$ (3)

Whence, $(r-1)s = ar^n - a$, or $s = \dfrac{ar^n - a}{r-1}$ (4)

By formula, Case I, $l = ar^{n-1}$; therefore, $rl = ar^n$.

Substituting rl for ar^n in the formula for s, the following formula is obtained·

$$s = \frac{rl - a}{r - 1}$$

EXAMPLES.

1. Find the sum of 10 terms of the series 2, 4, 8, etc.

PROCESS.

$$s = \frac{ar^n - a}{r - 1}$$

$$s = \frac{2 \times 2^{10} - 2}{2 - 1} = 2046$$

EXPLANATION.—In this example, $a = 2$, $r = 2$, $n = 10$. Substituting in the first formula obtained for the sum, the sum is 2046.

2. Find the sum of 11 terms of the series 1, 2, 4, 8, etc.

3. Find the sum of 9 terms of the geometrical series 1, 3, 9, 27, etc.

4. Find the sum of 12 terms of the geometrical series 4, 8, 16, 32, 64, etc.

5. Find the sum of 11 terms of the geometrical series 3, 9, 27, 81, 243, etc.

6. Find the sum of 10 terms of the geometrical series $2a$, $4a$, $8a$, etc.

7. Find the sum of 10 terms of the geometrical series $2x^2$, $6x^2$, $18x^2$, etc.

8. Find the sum of n terms of the series 2, 4, 8, 16, etc.

9. Find the sum of 10 terms of the series 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, etc.

10. Find the sum of 8 terms of the series 8, 2, $\frac{1}{2}$, $\frac{1}{8}$, etc.

11. The extremes of a geometrical progression are 4 and 1024, and the ratio 4. What is the sum of the series?

12. The extremes of a geometrical progression are 2 and 512, and the ratio 2. What is the sum of the series?

13. What is the sum of a series in which the first term is 2, the last term 0, and the ratio $\frac{1}{2}$; or what is the sum of the infinite series 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, etc.?

14. What is the sum of the infinite series 6, 3, $1\frac{1}{2}$, etc.; or what is the sum of a series in which the first term is 6, the last term 0, and the ratio $\frac{1}{2}$?

15. What is the sum of the infinite series 2, $\frac{2}{3}$, $\frac{2}{9}$, etc.?

16. What is the sum of the infinite series $1 + \frac{1}{x^2} + \frac{1}{x^4} + \frac{1}{x^6}$, etc.?

17. What is the sum of the infinite series $x - y + \frac{y^2}{x} - \frac{y^3}{x^2} + \frac{y^4}{x^3}$, etc.?

18. A man engaged to work for 8 months, upon condition that he should receive \$2 for the first month, \$4 for the second, \$8 for the third, and so on. How much did he earn in the time?

19. A man rented a farm of 500 acres for 20 years, agreeing to pay \$1 for the first year, \$2 for the second year, \$4 for the third year, and so on. What was the entire amount of rent paid for the farm?

349. Formulas for finding any element.

By combining the fundamental formulas given in the previous cases, all problems which may arise in Geometrical Progression may be solved.

When any three of the elements are given, the other two can be found.

Deduce the following formulas:

GIVEN.	REQUIRED.	FORMULAS.	
$a, r, n,$	$l, s.$	$l = ar^{n-1}.$	$s = \dfrac{ar^n - a}{r-1}.$
$l, r, n,$	$a, s.$	$a = \dfrac{l}{r^{n-1}}.$	$s = \dfrac{lr^n - l}{r^n - r^{n-1}}.$
$n, r, s,$	$a, l.$	$a = \dfrac{s(r-1)}{r^n - 1}.$	$l = \dfrac{(r-1)sr^{n-1}}{r^n - 1}.$
$a, l, n,$	$r, s.$	$r = \sqrt[n-1]{\dfrac{l}{a}}.$	$s = \dfrac{\sqrt[n-1]{l^n} - \sqrt[n-1]{a^n}}{\sqrt[n-1]{l} - \sqrt[n-1]{a}}.$
$a, n, s,$	$r, l.$	$ar^n - rs = a - s.$	$l(s-l)^{n-1} = a(s-a)^{n-1}.$
$l, n, s,$	$r, a.$	$(s-l)r^n + l = sr^{n-1}.$	$a(s-a)^{n-1} = l(s-l)^{n-1}.$
$a, r, l,$	$s, n.$	$s = \dfrac{rl - a}{r-1}.$	$n = \dfrac{\log. l - \log. a}{\log. r} + 1.$
$a, l, s,$	$r, n.$	$r = \dfrac{s-a}{s-l}.$	$n = \dfrac{\log. l - \log. a}{\log. (s-a) - \log. (s-l)} + 1.$
$a, r, s,$	$l, n.$	$l = \dfrac{a + (r-1)s}{r}.$	$n = \dfrac{\log. (a + (r-1)s) - \log. a}{\log. r}.$
$l, r, s,$	$a, n.$	$a = rl - (r-1)s.$	$n = \dfrac{\log. l - \log. (lr - (r-1)s)}{\log. r} + 1.$

The values of n are given here to complete the scheme. They may be found by the student after studying Logarithms.

SPECIAL APPLICATIONS.

350. 1. When x represents the first term and y the ratio, the series may be represented by

$$x,\ xy,\ xy^2,\ xy^3,\ xy^4,\ \text{etc.}$$

2. When there are *three* terms in the series, they may be represented as follows:

x^2, xy, y^2, in which $\dfrac{y}{x}$ represents the ratio; or, by

x, $\sqrt{xy}$, y, in which $\sqrt{\dfrac{y}{x}}$ represents the ratio.

3. When there are *four* terms in the series, they may be represented as follows:

$\dfrac{x^2}{y}$, x, y, and $\dfrac{y^2}{x}$, in which $\dfrac{y}{x}$ represents the ratio.

PROBLEMS.

1. The sum of three numbers in geometrical progression is 7, and the sum of their squares is 21. What are the numbers?

SOLUTION.

Let x, $\sqrt{xy}$, and y represent the numbers.

By the conditions, $\begin{cases} x + \sqrt{xy} + y = 7 & (1) \\ x^2 + xy + y^2 = 21 & (2) \end{cases}$

Dividing (2) by (1), $x - \sqrt{xy} + y = 3$ (3)

Adding (1) and (3), $2x + 2y = 10$ (4)

$x + y = 5$ (5)

Subtracting (5) from (1), $\sqrt{xy} = 2$ (6)

$xy = 4$ (7)

Whence, $x = 1$, $\sqrt{xy} = 2$, $y = 4$

2. The sum of a geometrical series containing 8 terms is 1785, and the ratio 2. What is the first term?

3. The sum of a geometrical series containing 6 terms is 1365, and the ratio 4. What is the first term?

4. The sum of a geometrical series is 1. The first term is $\frac{1}{3}$ and the last term 0. What is the ratio?

5. The first term of a geometrical series is 32, the last term 4000, and the number of terms 4. What is the ratio?

6. Find three terms in geometrical progression whose sum is 13, and the sum of whose squares is 91.

7. The product of three numbers in geometrical progression is 8, and the sum of their squares 21. What are the numbers?

8. The sum of the first and third of four numbers in geometrical progression is 10, and the sum of the second and fourth is 30. What are the numbers?

SUGGESTION.—Represent the numbers by x, xy, xy^2, and xy^3.

9. The sum of four numbers in geometrical progression is 15, and the last term divided by the sum of the means is $\frac{4}{3}$. What are the numbers?

10. The sum of three numbers in geometrical progression is 14, and the sum of the extremes multiplied by the mean is 40. What are the numbers?

11. The sum of the first two of four numbers in geometrical progression is 10, and the sum of the last two is $22\frac{1}{2}$. What are the numbers?

12. Find three numbers in geometrical progression such that the sum of the first and last is 20, and the square of the mean 36.

13. A man bought a farm for $5000, agreeing to pay principal and interest in five equal annual installments. What will be the annual payment, including interest at 6%?

SOLUTION.

Let $P =$ any principal.

$p =$ the annual payment.

$r =$ the rate per cent.

$P(1+r) =$ amount due at end of first year.

$P(1+r) - p =$ sum due after first payment is made.

$P(1+r)^2 - p(1+r) =$ amount due at end of second year.

$P(1+r)^2 - p(1+r) - p =$ sum due after second payment is made.

$P(1+r)^3 - p(1+r)^2 - p(1+r) - p =$ sum due after third payment is made.

$P(1+r)^5 - p(1+r)^4 - p(1+r)^3 - p(1+r)^2 - p(1+r) - p =$ sum due after fifth payment is made.

Since the debt was discharged when the fifth payment was made,

$$P(1+r)^5 - p(1+r)^4 - p(1+r)^3 - p(1+r)^2$$
$$p(1+r) - p = 0.$$

Whence, $p(1+r)^4 + p(1+r)^3 + p(1+r)^2 + p(1+r) + p = P(1+r)^5.$

$$p = \frac{P(1+r)^5}{(1+r)^4 + (1+r)^3 + (1+r)^2 + (1+r) + 1}.$$

Or, since the denominator forms a geometrical series,

$$p = \frac{P(1+r)^5}{\frac{(1+r)^5 - 1}{r}} = \frac{Pr(1+r)^5}{(1+r)^5 - 1}.$$

And, in general,

$$p = \frac{Pr(1+r)^n}{(1+r)^n - 1} = \frac{\$5000 \times .06 \times (1.06)^5}{(1.06)^5 - 1} = \$1186.98.$$

14. If a man agrees to pay a debt of \$3000, bearing interest at 7%, in 6 equal annual installments, what would be the annual payment?

LOGARITHMS.

351. 1. What is the value of 2^3? What power of 2 equals 8? What power of 2 equals 16? What power of 2 equals 32?

2. What power of 3 equals 9? What 27? What 81? What 243?

3. What power of 4 equals 4? What 16? What 64? What 256?

4. What power of 10 equals 10? What power of 10 equals 1? What 100? What 1000?

352. The **Logarithm** of a number is the index of the power to which a constant number must be raised to produce the given number.

Thus, when 4 is the constant number, 2 is the logarithm of 16, for $4^2 = 16$.

353. The constant number which must be raised to some power in order to produce the given numbers is called the **Base** of the logarithms.

354. Logarithms may be computed upon any base, but the base of the **Common System** of Logarithms is 10.

Since $10^0 = 1$, the logarithm of 1 is 0.
Since $10^1 = 10$, the logarithm of 10 is 1.
Since $10^2 = 100$, the logarithm of 100 is 2.
Since $10^3 = 1000$, the logarithm of 1000 is 3.
Since $10^{-1} = \frac{1}{10}$, the logarithm of .1 is -1.
Since $10^{-2} = \frac{1}{100}$, the logarithm of .01 is -2.
Since $10^{-3} = \frac{1}{1000}$, the logarithm of .001 is -3.

355. It is evident that the logarithm of any number between 1 and 10 is less than 1, and is a fraction; between 10 and 100, 1 plus a fraction; between 100 and 1000, 2 plus a fraction, etc.

356. The integral part of a logarithm is called the **Characteristic**; the fractional part, the **Mantissa.**

From the examples given in Art. 354, it follows that—

357. PRINCIPLES.—1. *The characteristic of the logarithm of an integral number is a number which is 1 less than the number of figures in the given number.*

2. *The characteristic of a decimal fraction is negative, and numerically one greater than the number of zeros immediately following the decimal point.*

Thus, the characteristic of 42 is 1; of 423 is 2; of 4234 is 3; of .01 is —2; of .42 is —1; of .324 is —1; of .00325 is —3.

It is evident that the *characteristic only* is negative, and, consequently, the mantissa is positive.

The sign of the characteristic is usually written above the characteristic.

358. The following examples will illustrate the *characteristic* and *mantissa*, and their significance:

Log. of $231.4 = 2.364363$, or $231.4 = 10^{2.364363}$.
Log. of $23.14 = 1.364363$, or $23.14 = 10^{1.364363}$.
Log. of $2.314 = 0.364363$, or $2.314 = 10^{\ .364363}$.
Log. of $.2314 = \bar{1}.364363$, or $.2314 = 10^{\bar{1}.364363}$.
Log. of $.02314 = \bar{2}.364363$, or $.02314 = 10^{\bar{2}.364363}$.

From an examination of the examples given, it is seen that in the logarithms of numbers expressed by the same figures, the decimal part, or *mantissa*, is the same, and the logarithms differ only in the *characteristic.* Hence, tables of logarithms of numbers contain only the *mantissas.*

TABLES OF LOGARITHMS.

359. The tables of logarithms on the next two pages give the *decimal part*, or *mantissa*, of the common logarithms of all numbers from 1 to 999 correct to five decimal places.

The logarithms given in the tables begin with the mantissa of 10, but since the mantissas of 10, 20, 30, 40, etc., are the same as the mantissas of 1, 2, 3, 4, etc., the table may be said to give the logarithms of numbers from 1 to 1000.

Explanation of Tables.

The left-hand column of each page of the table is a column of numbers. It is designated by **N.**

The mantissas of the logarithms of these numbers are opposite them in the next column.

At the top of each page and extending across the top are found the figures **0, 1, 2, 3, 4, 5, 6, 7, 8, 9,** each standing over a column of figures. These figures are the right-hand figures of numbers whose left-hand figures are given in the left-hand column, and the figures under them are the corresponding mantissas of the numbers.

It will be seen that the first column of mantissas contains five figures, while the others contain only four. This difference is due to the fact that the left-hand figure in the mantissas, which is usually the same for a whole horizontal column, is omitted except in the first column. When the first figure of the mantissa is 0, the left-hand figure of the mantissa for the rest of the numbers in that horizontal line, is one greater than the first figure of the left-hand column of mantissas in that same horizontal line.

By subtracting these mantissas, each from the one next succeeding, it is found that those in the same horizontal line have nearly the same difference.

This *Average Difference* is found in the column marked **D.**

Table of Common Logarithms.

N.	0	1	2	3	4	5	6	7	8	9	D.
10	00000	0432	0860	1284	1703	2119	2531	2938	3342	3743	414
11	04139	4532	4922	5308	5690	6070	6446	6819	7188	7555	378
12	07918	8279	8636	8991	9342	9691	0037	0380	0721	1059	348
13	11394	1727	2057	2385	2710	3033	3354	3672	3988	4301	322
14	14613	4922	5229	5534	5836	6137	6435	6732	7026	7319	300
15	17609	7898	8184	8469	8752	9033	9312	9590	9866	2140	280
16	20412	0683	0952	1219	1484	1748	2011	2272	2531	2789	263
17	23045	3300	3553	3805	4055	4304	4551	4797	5042	5285	248
18	25527	5768	6007	6245	6482	6717	6951	7184	7416	7646	235
19	27875	8103	8330	8556	8780	9003	9226	9447	9667	9885	223
20	30103	0320	0535	0750	0963	1175	1387	1597	1806	2015	212
21	32222	2428	2634	2838	3041	3244	3445	3646	3846	4044	202
22	34242	4439	4635	4830	5025	5218	5411	5603	5793	5984	193
23	36173	6361	6549	6736	6922	7107	7291	7475	7658	7840	185
24	38021	8202	8382	8561	8739	8917	9094	9270	9445	9620	177
25	39794	9967	0140	0312	0483	0654	0824	0993	1163	1330	170
26	41497	1664	1830	1996	2160	2325	2488	2651	2813	2975	164
27	43136	3297	3457	3616	3775	3933	4091	4248	4404	4560	158
28	44716	4871	5025	5179	5332	5484	5637	5788	5939	6090	152
29	46240	6389	6538	6687	6835	6982	7129	7276	7422	7567	147
30	47712	7857	8001	8144	8287	8430	8572	8714	8855	8996	142
31	49136	9276	9415	9554	9693	9831	9969	0106	0243	0379	138
32	50515	0651	0786	0920	1055	1188	1322	1455	1587	1720	134
33	51851	1983	2114	2244	2375	2504	2634	2763	2892	3020	130
34	53148	3275	3403	3529	3656	3782	3908	4033	4158	4283	126
35	54407	4531	4654	4777	4900	5023	5145	5267	5388	5509	122
36	55630	5751	5871	5991	6110	6229	6348	6467	6585	6703	119
37	56820	6937	7054	7177	7287	7403	7519	7634	7749	7864	116
38	57978	8092	8206	8320	8433	8546	8657	8771	8883	8995	113
39	59106	9218	9329	9439	5550	9660	9770	9879	9988	0097	110
40	60206	0314	0423	0531	0639	0746	0853	0959	1066	1172	107
41	61278	1384	1490	1595	1700	1805	1909	2014	2118	2221	105
42	62325	2428	2531	2634	2737	2839	2941	3043	3144	3246	102
43	63347	3448	3548	3649	3749	3849	3949	4048	4147	4246	100
44	64345	4444	4542	4640	4738	4836	4933	5031	5128	5225	98
45	65321	5418	5514	5610	5706	5801	5896	5992	6087	6181	95
46	66276	6370	6464	6558	6652	6745	6839	6932	7025	7117	93
47	67210	7302	7394	7486	7578	7669	7761	7852	7943	8034	91
48	68124	8215	8305	8395	8485	8574	8664	8753	8842	8931	90
49	69020	9108	9197	9285	9373	9461	9546	9636	9723	9810	88
50	69897	9984	0070	0157	0243	0329	0415	0501	0586	0672	86
51	70757	0842	0927	1012	1096	1181	1265	1349	1433	1517	84
52	71600	1684	1767	1850	1933	2016	2099	2181	2263	2346	83
53	72428	2509	2591	2673	2754	2835	2916	2997	3078	3159	81
54	73239	3320	3400	3480	3560	3640	3719	3799	3878	3957	80

Table of Common Logarithms.

N.	0	1	2	3	4	5	6	7	8	9	D.
55	74036	4115	4194	4273	4351	4429	4507	4586	4663	4741	78
56	74819	4896	4974	5051	5128	5205	5282	5358	5435	5511	77
57	75587	5664	5740	5815	5891	5967	6042	6118	6193	6268	76
58	76343	6418	6492	6567	6641	6716	6790	6864	6938	7012	74
59	77085	7159	7232	7305	7379	7452	7525	7591	7670	7743	73
60	77815	7887	7960	8032	8104	8176	8247	8319	8390	8462	72
61	78533	8604	8675	8746	8817	8888	8958	9029	9099	9169	71
62	79239	9309	9379	9449	9518	9588	9657	9727	9796	9865	69
63	79934	0003	0072	0140	0209	0277	0346	0414	0482	0550	68
64	80618	0686	0754	0821	0889	0956	1023	1090	1158	1224	67
65	81291	1358	1425	1491	1558	1624	1690	1757	1823	1889	66
66	81954	2020	2086	2151	2217	2282	2347	2413	2478	2543	65
67	82607	2672	2737	2802	2866	2930	2995	3059	3123	3187	64
68	83251	3315	3378	3442	3506	3569	3632	3696	3759	3822	63
69	83885	3948	4011	4073	4136	4198	4261	4323	4386	4448	62
70	84510	4572	4634	4696	4757	4819	4880	4942	5003	5065	62
71	85126	5187	5248	5309	5370	5431	5491	5552	5612	5673	61
72	85733	5794	5854	5914	5974	6034	6094	6153	6213	6273	60
73	86332	6392	6451	6510	6570	6629	6688	6747	6806	6864	59
74	86923	6982	7040	7099	7157	7216	7274	7332	7390	7448	58
75	87506	7564	7622	7679	7737	7795	7852	7910	7967	8024	58
76	88081	8138	8195	8252	8309	8366	8423	8480	8536	8593	57
77	88649	8705	8762	8818	8874	8930	8986	9042	9098	9154	56
78	89209	9265	9321	9376	9432	9487	9542	9597	9653	9708	55
79	89763	9818	9873	9927	9982	0037	0091	0146	0200	0255	55
80	90309	0363	0417	0472	0526	0580	0634	0687	0741	0795	54
81	90849	0902	0956	1009	1062	1116	1169	1222	1275	1328	53
82	91381	1434	1487	1540	1593	1645	1698	1751	1803	1855	53
83	91908	1960	2012	2065	2117	2169	2221	2273	2324	2376	52
84	92428	2480	2531	2583	2634	2686	2737	2788	2840	2891	51
85	92942	2993	3044	3095	3146	3197	3247	3298	3349	3399	51
86	93450	3500	3551	3601	3651	3702	3752	3802	3852	3902	50
87	93952	4002	4052	4101	4151	4201	4250	4300	4349	4399	50
88	94448	4498	4547	4596	4645	4694	4743	4792	4841	4890	49
89	94939	4988	5036	5085	5134	5182	5231	5279	5328	5376	49
90	95424	5472	5521	5569	5617	5665	5713	5761	5809	5856	48
91	95904	5952	5999	6047	6095	6142	6190	6237	6284	6332	47
92	96379	6426	6473	6520	6567	6614	6661	6708	6755	6802	47
93	96848	6895	6942	6988	7035	7081	7128	7174	7220	7267	46
94	97313	7359	7405	7451	7497	7543	7589	7635	7681	7727	46
95	97772	7818	7864	7909	7955	8000	8046	8091	8137	8182	45
96	98227	8272	8318	8363	8408	8453	8498	8543	8588	8632	45
97	98677	8722	8767	8811	8856	8900	8945	8989	9074	9078	45
98	99123	9167	9211	9255	9300	9346	9388	9432	9476	9520	44
99	99564	9607	9651	9695	9799	9782	9826	9870	9913	9957	44

360. To find the Logarithm of a Number.

1. Find the logarithm of 3824.

EXPLANATION.—Since the tables on the preceding pages contain the mantissas of no numbers expressed by more than three figures, the mantissa of 382 is first found, which is the same as the mantissa of 3820. It is found to be .58206.

Since the mantissa of the next larger number, 383 or 3830, is 114 hundred-thousandths greater than the mantissa of 3820, every unit added to 3820 will add .1 of 114 hundred-thousandths to the mantissa, and 4 will add .4 of 114 hundred-thousandths, or 45 hundred-thousandths. This added to .58206 gives .58251, the mantissa of 3824.

Since the number is expressed by 4 figures, the characteristic is 3. Therefore, the logarithm of 3824 is 3.58251. Or,

From the table, the decimal part of the logarithm of the first three figures, 382, is .58206; the average difference, 113 multiplied by .4, the remaining part of the number, gives 45, which, added to the right-hand figures of the decimal part already found, gives .58251.

Since the number is expressed by 4 figures, the characteristic is 3. Therefore, log. of 3824 = 3.58251.

2. Find the logarithm of 318.
3. Find the logarithm of 285.
4. Find the logarithm of 486.
5. Find the logarithm of 335.
6. Find the logarithm of 33.6.
7. Find the logarithm of 2.68.
8. Find the logarithm of .384.
9. Find the logarithm of 4831.
10. Find the logarithm of 3846.
11. Find the logarithm of 2785.
12. Find the logarithm of 3169.
13. Find the logarithm of 1875.
14. Find the logarithm of 2.345.
15. Find the logarithm of 1.684.

361. To find a Number whose Logarithm is given.

1. Find the number whose logarithm is 3.95323.

PROCESS.

Given log.,	3.95323
Log. next less,	3.95279
Difference of logs.,	44
Tabular difference,	49

$$44 \div 49 = .89 +$$

Number corresponding to mantissa .95279 is 897.

Annexing to 897 the rest of number, .89 +, the whole number is 8978.9 +, since it is expressed by four figures.

EXPLANATION.—The logarithm nearest the given logarithm, and next less, is 3.95279. This, subtracted from the given logarithm, gives 44 as a remainder. By referring to the average difference column in the table, the difference is found to be 49, and 44 divided by 49 gives .89 +. The number corresponding to the logarithm 3.95279, consists of 4 integral figures, the first 3 of which are found from the table to be 897. Annexing the part found by dividing the difference of the logarithms by the average difference, the number is 8978.9 +.

2. Find the number whose logarithm is 2.38257.
3. Find the number whose logarithm is 2.18625.
4. Find the number whose logarithm is 0.23146.
5. Find the number whose logarithm is $\bar{1}.28643$.
6. Find the number whose logarithm is $\bar{2}.98465$.
7. Find the number whose logarithm is 3.18425.
8. Find the number whose logarithm is 2.86435.
9. Find the number whose logarithm is $\bar{3}.24685$.
10. Find the number whose logarithm is 2.98456.

362. Multiplication by Logarithms.

Since logarithms are the exponents of the powers to which a constant quantity is to be raised, how may quantities be multiplied when their logarithms are known?

1. Multiply 32.4 by 26.

PROCESS.

$$\begin{array}{ll} \text{Log. of } 32.4 &= 1.51055 \\ \text{Log. of } 26 &= 1.41497 \\ \hline \text{Sum of logs.} &= 2.92552 \end{array}$$

2.92552 is log. of 842.4.

Therefore, $32.4 \times 26 = 842.4$.

EXPLANATION.—We find the logarithm of each of the given numbers, and, inasmuch as the logarithms are exponents of a constant quantity, the product of these numbers will be the constant quantity, with an exponent equal to the sum of the exponents of this constant quantity. The sum of these exponents or logarithms is 2.92552. The number corresponding to this logarithm is 842.4, the product of the numbers.

2. Multiply 2.3 by 3.7.
3. Multiply 25 by 3.5.
4. Multiply 216 by 3.5.
5. Multiply 312 by .24.
6. Multiply 123 by 3.4.
7. Multiply 2.24 by 2.6.
8. Multiply .0023 by .26.
9. Multiply .0015 by .015.

363. Division by Logarithms.

Since in multiplication we add the logarithms, or the exponents, of the constant quantity, how may division be performed?

1. Divide .05475 by 15.

PROCESS.

$$\begin{array}{ll} \text{Log. of } .05475 \text{ is } & \bar{2}.73839 \\ \text{Log. of } 15 \text{ is } & 1.17609 \\ \hline \text{Difference of logs. is } & \bar{3}.56230 \end{array}$$

$\bar{3}.56230$ is log. of .00365.

Therefore, $.05475 \div 15 = .00365$.

EXPLANATION.—We find the logarithm of each number, and then subtract the logarithm of the divisor from that of the dividend. The number corresponding to this difference between the logarithms will be the quotient.

2. Divide 2.45 by 9.8.
3. Divide 18.312 by 24.
4. Divide 105.7 by 3.5.
5. Divide 135.05 by .037.
6. Divide .04905 by .327.
7. Divide 34.43 by .011.
8. Divide 259.2 by .012.
9. Divide 87.36 by 2.1.
10. Divide 97.24 by .022.
11. Divide 13.696 by 32.

364. Involution by Logarithms.

Since logarithms are exponents, how may quantities, whose logarithms are known, be raised to any power?

1. What is the second power of 25?

PROCESS.

$$\begin{array}{lr} \text{Log. of 25 is} & 1.39794 \\ & \underline{\quad\quad 2} \\ \text{Log. of the power is} & 2.79588 \end{array}$$

2.79588 is log. of 625.
Therefore, $(25)^2 = 625$.

EXPLANATION.—Since in involution we multiply the exponent of the quantity by the exponent of the power to which it is to be raised, in involution by logarithms we may find the logarithm of the given quantity, and multiply it by the exponent of the power to which it is to be raised; the number corresponding to the resulting logarithm will be the power sought.

2. What is the second power of 19?
3. What is the second power of 35?
4. What is the second power of 45?
5. What is the second power of 29?
6. What is the third power of 32?
7. What is the third power of 25?
8. What is the third power of 14?

365. Evolution by Logarithms.

Since in involution the logarithms, or exponents, are multiplied to produce the power, what must be done when roots are to be extracted?

1. What is the square root of 625?

PROCESS.

Log. of 625 is 2.79588
Dividing by 2, 1.39794
1.39794 is the log. of 25.
Therefore, $(625)^{\frac{1}{2}} = 25$.

EXPLANATION.—Since in evolution we divide the exponent of the quantity by the number corresponding to the root to be extracted, in evolution by logarithms we find the logarithm of the given number, and divide it by the index of the required root; the number corresponding to the resulting logarithm will be the root sought.

2. What is the square root of 196?
3. What is the square root of 256?
4. What is the square root of 4096?
5. What is the square root of 1296?
6. What is the cube root of 4096?
7. What is the cube root of 13824?
8. What is the cube root of 74088?

MISCELLANEOUS EXAMPLES.

366. 1. Add $3x^{\frac{1}{2}}y - 4x\sqrt{y} + 5$, $\sqrt{xy} + 2xy^{\frac{1}{2}} + 4$, $6y\sqrt{x} - \sqrt{xy} - 7$, $4y\sqrt{x} - 3y^{\frac{1}{2}}x - 6$, and $2 + 5xy^{\frac{1}{2}} - 3yx^{\frac{1}{2}}$.

2. Add $2x^4y^2 + 2x^{-n}y^{\frac{3}{2}} - 3x^2$, $2b^2x^{-m}y^{\frac{3}{2}} - a + 6x^2$, $3x^4y^2 - 5cx^4y^2 - 2b^2x^{-m}y^{\frac{3}{2}} + 3a$, and $2x^3 + cy$.

3. Add $4b - 2cy^{-\frac{2}{3}} + m$ to $7cy^{-\frac{2}{3}} + 8ax - 5b + 10ax - 2b + 8m - 3$, and subtract from the result the sum of $5ax - 4m + 3$, $5cy^{-\frac{2}{3}} - 3ax - 6$, and $3m - 10cy^{-\frac{2}{3}} - 2m$.

4. From $17xy^2 + 3xz + 10a$ subtract $7xy^2 + 4xz + 12x - 2bx$.

5. Multiply $a^4 + a^3b + a^2b^2 + ab^3 + b^4$ by $a - b$.

6. Multiply $3x^{m-1} - 2y^{n-2}$ by $2x - 3y^2$.

7. Multiply $3x^{-\frac{1}{2}} - 2y^{\frac{2}{3}}$ by $3x^{-\frac{1}{2}} + 2y^{\frac{2}{3}}$.

8. Multiply $2x^{\frac{m}{2n}} - 3y^{\frac{m-n}{2}}$ by $2x^{\frac{m}{2n}} + 3y^{\frac{m-n}{2}}$.

9. Expand $(x^{2n} + y^{2m})(x^{2n} + y^{2m})$

10. Divide $x^4 - y^4$ by $x + y$.

11. Divide $x^7 + y^7$ by $x + y$.

12. Divide $x^n - y^n$ by $x - y$ to 6 terms.

13. Divide $\frac{x^3}{2} + x^2 + \frac{3x}{8} + \frac{3}{4}$ by $\frac{x}{2} + 1$.

14. Factor $4x^2 + 4xy + y^2$.

15. Factor $x^4 - y^4$.

16. Factor $x^2 - 2x - 35$.

17. Factor $x^2 - 6x - 27$.

18. Factor $x^8 - y^8$.

19. Find the greatest common divisor of $x^2 - y^2$, $x^2 - 2xy + y^2$, and $xy - y^2$.

20. Find the greatest common divisor of $6x^4 + 11x^2 + 3$ and $2x^4 - 5x^2 - 12$.

21. Find the greatest common divisor of $4x^4 - 24x^3 + 34x^2 + 12x - 18$ and $4x^3 - 18x^2 + 19x - 3$.

22. Find the greatest common divisor of $x^4 - 4x^3 - 16x^2 + 7x + 24$ and $2x^3 - 15x^2 + 9x + 40$.

23. Find the least common multiple of $2a^2x$, $3axy$, and $4a^2x^2y^3$.

24. Find the least common multiple of $x^2 - y^2$, $x + y$, $x - y$, and $xy - y^2$.

25. Reduce $\dfrac{x^2 - 7x + 10}{2x^2 - x - 6}$ to its lowest terms.

26. Reduce $\dfrac{x^3 - 4x^2 + 9x - 10}{x^3 + 2x^2 - 3x + 20}$ to its lowest terms.

27. Reduce $\frac{a^2 - b^2 - 2bc - c^2}{a^2 + 2ab + b^2 - c^2}$ to its lowest terms.

28. Simplify $\frac{x}{x+1} - \frac{x}{1-x} + \frac{x^2}{x^2-1}$.

29. Simplify $\frac{3+2x}{2-x} - \frac{2-3x}{2+x} + \frac{16x - x^2}{x^2-4}$.

30. Simplify $\frac{1}{(a-b)(b-c)} + \frac{1}{(b-a)(a-c)} - \frac{1}{(c-a)(c-b)}$.

31. Simplify $\frac{1+x}{1+x+x^2} + \frac{1-x}{1-x+x^2} - \frac{2}{1+x^2+x^4}$.

32. Simplify $\left(x + \frac{1}{x}\right)\left(x^2 + \frac{1}{x^2}\right)\left(x - \frac{1}{x}\right)$.

33. Simplify $\frac{a+b}{a-b} + \frac{a-b}{a+b} - \frac{2(a^2-b^2)}{a^2+b^2}$.

34. Simplify $\left(\frac{x}{1+\frac{1}{x}} + 1 - \frac{1}{x+1}\right) \div \left(\frac{x}{1-\frac{1}{x}} - x - \frac{1}{x-1}\right)$.

35. Divide $\left(\frac{x}{x-y} - \frac{y}{x+y}\right)$ by $\left(\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2-y^2}\right)$.

36. Multiply $\frac{x^2 - 2xy + y^2 - z^2}{x^2 + 2xy + y^2 - z^2}$ by $\frac{x+y-z}{x-y+z}$.

37. Simplify $\frac{2x - 3 + \frac{1}{x}}{\frac{2x-1}{x}}$.

38. Divide $2 - \frac{\sqrt{x}}{\sqrt{y}} - \frac{\sqrt{y}}{\sqrt{x}}$ by $\frac{\sqrt{x} - \sqrt{y}}{\sqrt{xy}}$.

39. Raise $x + y$ to the seventh power.

40. Expand $(2a + 3b)^5$.

41. Expand $(a + b)^{10}$.

42. Raise $\sqrt{x + y}$ to the sixth power.

43. Find the sum of $\sqrt[3]{x^7y}$, $\sqrt[3]{8x^4y^4}$, $\sqrt[3]{xy^7}$.

44. From $\sqrt{3x^2z + 6xyz + 3y^2z}$ subtract $\sqrt{12y^2z}$.

45. From $6\sqrt[3]{32}$ subtract $6\sqrt[3]{\frac{4}{27}}$

46. Multiply $\sqrt{x} - \sqrt{y}$ by $\sqrt{x} + \sqrt{y}$.

47. Multiply $\sqrt{a} + \sqrt{b}$ by $\sqrt{a} + \sqrt{b}$.

48. Given $7 - (7 + 7 - (7 + x)) = 7$, to find the value of x.

49. Given $a - \frac{m + n}{x} = b - \frac{m - n}{x}$, to find the value of x.

50. Given $\frac{1}{a - b} + \frac{a - b}{x} = \frac{1}{a + b} + \frac{a + b}{x}$, to find the value of x.

51. Given $\frac{bx}{a} - \frac{d}{c} = \frac{a}{b} - \frac{cx}{d}$, to find the value of x.

52. Given $\frac{4x + 3}{9} + \frac{7x - 29}{5x - 12} = \frac{8x + 19}{18}$, to find the value of x.

53. Given $(x + \frac{5}{2})(x - \frac{3}{2}) - (x + 5)(x - 3) + \frac{3}{4} = 0$, to find the value of x.

54. Given $\frac{m(x + a)}{x + b} + \frac{n(x + b)}{x + a} = m + n$, to find the value of x.

55. Given $\frac{x^2 - 16}{x + 4} = 6 - x$, to find the value of x.

56. Given $(1+x)^5+(1-x)^5=242$, to find x.

57. Given $\sqrt{4+x}=\dfrac{3}{\sqrt{4-x}}$, to find the value of x.

58. Given $\sqrt{x-32}=16-\sqrt{x}$, to find the value of x.

59. Given $\sqrt{4x+21}=2\sqrt{x}+1$, to find the value of x.

60. Given $\dfrac{\sqrt{9x}-4}{\sqrt{x}+2}=\dfrac{15+3\sqrt{x}}{\sqrt{x}+40}$, to find the value of x.

61. Given $\sqrt{2+x}+\sqrt{x}=\dfrac{4}{\sqrt{2+x}}$, to find the value of x.

62. Given $\left\{\begin{matrix} x+y=7 \\ x+z=8 \\ y+z=9 \end{matrix}\right\}$, to find the values of x, y, and z.

63. Given $\left\{\begin{matrix} \frac{1}{2}x+\frac{1}{3}y+\frac{1}{4}z=22 \\ \frac{1}{4}x+y+\frac{1}{2}z=33 \\ x+\frac{1}{4}y+\frac{1}{8}z=19 \end{matrix}\right\}$, to find the values of x, y, and z.

64. Given $\left\{\begin{matrix} 7x-2z+3w=17 \\ 4y-2z+v=11 \\ 5y-3x-2w=8 \\ 4y-3w+2v=9 \\ 3z+8w=33 \end{matrix}\right\}$, to find the values of the unknown quantities.

65. Given $\left\{\begin{matrix} cx+y+az=a+ac+c \\ c^2x+y+a^2z=3ac \\ acx+2y+acz=a^2+2ac+c^2 \end{matrix}\right\}$, to find the values of the unknown quantities.

66. Given $\begin{cases} x^2 + xy = 12 \\ y^2 + xy = 24 \end{cases}$, to find the values of x and y.

67. Given $\begin{cases} 3x + 3y = 15 \\ xy = 6 \end{cases}$, to find the values of x and y.

68. Given $\begin{cases} x^2y + xy^2 = 180 \\ x^3 + y^3 = 189 \end{cases}$, to find the values of x and y.

69. Given $\begin{cases} x - y = 8(\sqrt{x} - \sqrt{y}) \\ \sqrt{xy} = 15 \end{cases}$, to find the values of x and y.

70. Given $\begin{cases} x^3 - y^3 = 56 \\ x - y = \dfrac{16}{xy} \end{cases}$, to find the values of x and y.

71. Given $\begin{cases} x^4 - y^4 = 369 \\ x^2 - y^2 = 9 \end{cases}$, to find the values of x and y.

72. Given $\begin{cases} x^2 + 2y^2 = 41 \\ x^2 + 2xy = 33 \end{cases}$, to find the values of x and y.

73. Given $\begin{cases} x^2 + x + y = 18 - y^2 \\ xy = 6 \end{cases}$, to find the values of x and y.

74. Given $\begin{cases} xy + xy^2 = 12 \\ x + xy^3 = 18 \end{cases}$, to find the values of x and y.

75. Given $\begin{cases} x^2y^2 = 96 - 4xy \\ x + y = 6 \end{cases}$, to find the values of x and y.

76. Given $\left\{\begin{array}{l} x+y+\sqrt{x+y}= 12 \\ \qquad x^3+y^3 \qquad = 189 \end{array}\right\}$, to find the values of x and y.

77. Given $\left\{\begin{array}{l} x^{\frac{3}{2}}+y^{\frac{2}{3}}=3x \\ x^{\frac{1}{2}}+y^{\frac{1}{3}}=x \end{array}\right\}$, to find the values of x and y.

78. Given $\left\{\begin{array}{r} 2xy-24(x+y)=-240 \\ x^2+y^2= 100 \end{array}\right\}$, to find the values of x and y.

79. Given $\left\{\begin{array}{r} (x-y)(x^2+y^2)=13 \\ x^2y-xy^2=6 \end{array}\right\}$, to find the values of x and y.

80. What two numbers, which are to each other as 3 to 4, have a product which is equal to twelve times their sum?

81. A person being asked the time of day, replied that the time past noon was equal to $\frac{4}{5}$ of the time to midnight. What was the time of day?

82. Find a number which being increased by 4 and the sum multiplied by 3, gives the same result as if half the number were multiplied by 8 and the product were diminished by 8.

83. Find two numbers in the proportion of 3 to 4 such that if 9 be added to each the sums will be as 6 to 7.

84. The sum of two numbers is 12, and the difference of their squares is 72. What are the numbers?

85. It is required to divide 99 into five such parts that the first may exceed the second by 3, may be less than the third by 10, greater than the fourth by 9, and less than the fifth by 16.

86. There are two numbers whose product is 6, and whose sum added to the sum of their squares is 18. What are the numbers?

87. What number is that to which if 12 be added, and from $\frac{1}{12}$ of the sum 12 be subtracted, the remainder will be 12?

88. A boy paid 20 cents for 200 apples and pears together, buying 25 apples for a cent and 25 pears for 3 cents. How many of each did he buy?

89. A steamboat, whose rate in still water is 10 miles per hour, descends a river whose velocity is 4 miles per hour, and returns. She was away for 10 hours. How far did she go?

90. Three years ago A's age was $\frac{1}{2}$ of B's, and 9 years hence it will be $\frac{5}{8}$ of it. What is the age of each?

91. There is a number whose three digits are the same; and if 4 times the sum of the digits is subtracted from the number, the remainder is 297. What is the number?

92. A woman being asked what she paid for her eggs, replied, "Six dozen cost as many cents as I can buy eggs for 32 cents." What was the price per dozen?

93. What fraction is that which will be doubled if the numerator is multiplied by 4 and 3 is added to the denominator; but will be halved if 2 is added to the numerator and the denominator is multiplied by 4?

94. The stones which paved a square court-yard would just cover a rectangular surface whose length was 6 yards longer and whose breadth was 4 yards shorter than the side of the square. What was the area of the court?

95. A gentleman had not room in his stables for 8 of his horses, so he built an additional stable one-half the size of the other, when he had room for 8 horses more than he had. How many horses had he?

96. A gentleman purchased two square lots of ground for \$300. Each of them cost as many cents per square rod as there were rods in a side of the other, and the sum of

the perimeters of both was 200 rods. What was the cost of each?

97. A gentleman who had a square lot of ground, reserved 10 square rods out of it, and sold the rest for $432, which was as many dollars per square rod as there were rods in the side of the original lot. What was the length of its side?

98. A and B hired a pasture, into which A put 4 horses, and B as many as cost him 18 shillings a week. Afterward B put in 2 additional horses, and found that he must pay 20 shillings per week. What was paid for the pasture per week?

99. The sum of two numbers is 40. If the greater is multiplied by 2, and the less by 3, the difference of the products will be 15. What are the numbers?

100. A general having lost a battle, found that he had only 3600 more than half his army left fit for action, 600 more than $\frac{1}{8}$ of his men being disabled by wounds, and the rest, which were $\frac{1}{5}$ of the whole army, being killed or taken prisoners. How many men had he in the army?

101. Four places are situated in the order of the letters, A, B, C, D. The distance from A to D is 34 miles; the distance from A to B is to the distance from C to D as 2 is to 3, and $\frac{1}{4}$ of the distance from A to B added to $\frac{1}{2}$ the distance from C to D is three times the distance from B to C. What are the respective distances?

102. Given $x+\sqrt{3}=\dfrac{2\sqrt{2}}{\sqrt{3}-x}$, to find the values of x.

103. Several persons incurred an expense of $12, which they were to share equally. If there had been 4 more in the company, the expense to each person would have been 50 cents less than it was. How many persons were there in the company?

104. It is between 11 and 12 o'clock, and the hour-hand and minute-hand make a straight line. What is the time?

105. A rectangular field, whose sides are to each other as 2 to 5, contains 4 acres. What is the length and breadth of the field?

106. Divide 18 into two such parts that the squares of those parts may be to each other as 25 to 16.

107. What will be the payment which will discharge a debt of $2000 in four years, paying principal and interest in equal annual installments, interest at 6%?

108. A rectangular plat of ground has a walk 6 feet wide around the outside, which contains $\frac{1}{4}$ as much area as the plat itself. If the sides are to each other as 3 to 4, what is the length and breadth of the plat?

109. Given $\left\{\begin{array}{l} x+y=24 \\ xy : x^2+y^2 :: 3 : 10 \end{array}\right\}$, to find x and y.

110. Given $\left\{\begin{array}{l} xy=320 \\ x^3-y^3 : (x-y)^3 :: 61 : 1 \end{array}\right\}$, to find x and y.

111. There are four numbers in arithmetical progression such that the sum of the two least is 20, and the sum of the two greatest is 44. What are the numbers?

112. A farmer has two cubical granaries. The side of one is 3 yards longer than the side of the other, and the difference in their solid contents is 117 cubic yards. What is the side of each?

113. A merchant expended a sum of money in goods, which he sold for $56, and gained a per cent. equal to the number of dollars which the goods cost him. How much did they cost him?

114. The sum of three numbers in geometrical progression is 13, and the sum of the extremes multiplied by the mean is 30. What are the numbers?

115. Given $\left\{\begin{array}{l} x^2 + xy = 12 \\ xy - 2y^2 = 1 \end{array}\right\}$, to find x and y.

116. There are two rectangular boxes, one containing 20 cubic feet more than the other. Their bases are squares, the sides of each being equal to the depth of the other. If the capacities of the boxes are in the ratio of 4 to 5, what is the depth of each box?

117. What three numbers in geometrical progression are there whose sum is 14, and the sum of whose squares is 84?

118. What is the square root of $a^2x^4 + b^2x^2 + c^2 + 2abx^3 + 2acx^2 + 2bcx$?

119. A merchant has three pieces of cloth whose lengths are in geometrical progression. The aggregate length of the three pieces is 70 yards, and the longest piece is 30 yards longer than the shortest. What is the length of each?

120. A father divided $2100 among his three sons, so that the shares were in geometrical progression, and the second had $300 more than the third. What was the share of each?

121. A vintner has two casks of wine, from each of which he draws 6 gallons, when he finds the quantities left are to each other as 4 to 7. He then puts into the less 3 gallons, and into the greater 4 gallons, when the quantities they contain are to each other as 7 to 12. How many gallons were there in each at first?

122. Some smugglers discovered a cave which would exactly hold their cargo, which consisted of 13 bales of cotton and 33 casks of wine. While they were unloading, a revenue cutter hove in sight, when they sailed away with 9 casks and 5 bales, leaving the cave two thirds full. How many bales, or how many casks, would the cave hold?

123. A farmer sold a meadow at such a rate that the price per acre was to the number of acres as 2 to 3. If he had received \$270 more for it, the price per acre would have been to the number of acres as 3 to 2. How many acres did he sell, and at what price per acre?

124. Given $\sqrt{x^5} - \frac{40}{\sqrt{x}} = 3x$, to find x.

125. The sum of two numbers is to their difference as 4 to 1, and the sum of their cubes is 152. What are the numbers?

126. A and B set out from two towns which were 204 miles apart, and traveled in a direct line until they met. A traveled 8 miles per hour; and the number of hours before they met was greater by 3 than the number of miles B traveled per hour. How far did each travel?

127. A merchant bought a number of pieces of cloth for \$225, which he sold at \$16 a piece, and gained by the sale as much as one piece cost him. How many pieces were there?

128. There are three numbers in arithmetical progression whose sum is 15. If 1, 4, and 19 be added to them respectively, they will be in geometrical progression. What are the numbers?

129. A and B agreed to reap a field of grain for 90 shillings. A could reap it in 9 days, and they promised to complete it in 5 days. They were obliged, however, to call to their assistance C, an inferior workman, who worked the last two days, in consequence of which B received 3s. 9d. less than he otherwise would have received. In what time could B and C reap the field?

130. Find two quantities such that their sum, their product, and the sum of their squares shall be equal to each other.

131. Find two quantities such that their product shall be equal to the difference of their squares, and the sum of their squares shall be equal to the difference of their cubes.

132. A sets out from London to York, and B, at the same time, from York to London, both traveling uniformly. A reaches York 25 hours and B reaches London 36 hours after they have met on the road. In what time did they each perform the journey?

133. From two towns, which were 102 miles apart, two persons, A and B, set out to meet each other. A traveled 3 miles the first day, 5 miles the second day, 7 miles the third day, and so on. B traveled 4 miles the first day, 6 the next, 8 the next, and so on. In how many days did they meet?

134. Given $x^4 - 2x^3 + x = 30$, to find x.

Resolve into factors by partially extracting the square root and factoring the remainder.

135. Given $x^3 - 6x^2 + 11x = 6$, to find x.

Multiply both members of the equation by x, and resolve into factors by extracting the square root partially and factoring the remainder.

136. Given $\left\{\begin{array}{l} x^2 + y^2 + 4\sqrt{x^2 + y^2} = 45 \\ x^4 + y^4 = 337 \end{array}\right\}$, to find x and y

137. Given $\left\{\begin{array}{r} x^4 + y^4 = 17 \\ x^3y + xy^3 = 10 \end{array}\right\}$, to find x and y.

138. A railway train, after traveling 2 hours, is detained by an accident 1 hour. It then proceeds, for the rest of the distance, at $\frac{3}{5}$ of its former rate, and arrives $7\frac{2}{3}$ hours behind time. If the accident had occurred 50 miles further on, the train would have arrived $6\frac{1}{3}$ hours behind time. What was the whole distance traveled by the train?

TEST QUESTIONS.

367. What is the difference between the arithmetical and the algebraic solution of a problem? Illustrate by the solution of a problem. What is an equation? What is a problem? What is a solution of a problem? What is a statement of a problem?

Define quantity. What are used to express quantity? How is the word quantity used in algebra? What are known quantities? How are they represented? What are unknown quantities? How are they represented? Since the value of neither a nor x is known, what is the propriety in calling a a known quantity and x an unknown quantity?

Define algebra. What is the sign of addition? What is it called? What is the sign of subtraction? What does it show? What are the signs of multiplication? Illustrate the use of each. What is the sign of division? In what other way may division be indicated? What is the sign of equality? What is formed when it is written between two equal expressions? What are the signs of aggregation? Illustrate their use.

What is the sign of involution? What is it called? Illustrate its use. When no exponent is written, what is the exponent? What is a power of a quantity? Illustrate the powers of numbers and literal quantities. How are powers named? What other name is given to the second power? To the third power?

What is a root of a quantity? Illustrate the roots of numbers and literal quantities. How are roots named? What other name is given to the second root? To the third root?

What is the sign of evolution? Illustrate its use. What is the index of a root? When no index is written at the opening of the radical sign, what root is indicated? What is the ambiguous sign? What is a coëfficient? What are the various kinds of coëfficients? Illustrate the use of each. When no coëfficient is expressed, what is the coëfficient?

What is an algebraic expression? What are the terms of an algebraic expression? What is a positive term? When the first term of an expression has no sign written, what sign is it understood to have? What is a negative term? What are similar terms? Illustrate them. What are dissimilar terms? What is a monomial? Illustrate. What is a polynomial? What is a binomial? What is a trinomial?

Define addition. Define sum. State the principles of addition. Illustrate their application. Give the cases in addition. Illustrate each by the solution of an appropriate example. Give the rule for addition. How may dissimilar terms be added when they have a common factor?

Define subtraction; minuend; subtrahend; difference, or remainder. What are the principles of subtraction? Illustrate their application. What are the cases in subtraction? What is the rule for subtraction? Show the truth of principles (1) and (2). How may dissimilar terms, which have a common factor, be subtracted? Give the principles relating to the use of the parenthesis. Illustrate their application.

What are the members of an equation? Which is the first member? The second? Define transposition. What is an axiom? Give five axioms and illustrate their truth. What is the principle relating to the transposition of quantities? What is the rule for the solution of equations that require transposition? What is meant by verifying a result? How may a result be verified? If the same quantity with the same sign is found on opposite sides of an equation, what may be done? What is the effect upon an equation if the signs of all the terms are changed at the same time?

Define multiplication; multiplicand; multiplier; product; factors of the product. What are the signs of multiplication? Illustrate their use. What are the principles relating to multiplication? Show the truth of principles (2) and (4). What are the cases in multiplication? What is the rule for Case I? What is the rule for Case II? Solve an example and explain the solution. What is it to expand an expression?

What is the square of the sum of two quantities? Illustrate. What is the square of the difference of two quantities? Illustrate. What is the product of the sum and difference of two quantities?

Illustrate. What is the product of two binomial quantities having a common term? Illustrate.

Define division; dividend; divisor; quotient; remainder. Give the principles of division. Show the truth of principles (1) and (3). Deduce the law of signs in division from the law of signs in multiplication. What is Case I? Solve an example under Case I. What is the rule? When an equal factor is found in both dividend and divisor, what may be done with it? What is Case II? Solve an example under Case II, explain the solution, and deduce a rule.

What are the principles relating to quantities having zero for an exponent, and to those having negative exponents? Develop the principles. Solve an example illustrating each principle.

Define an exact divisor; factors; a prime quantity; a prime factor; factoring. What is Case I in factoring? Solve an example under this case. Give the rule. What is Case II in factoring? Solve an example under Case II. Give the rule. What is Case III? Solve an example. Give an explanation of the process. Give the rule. What is Case IV in factoring? Solve an example. Give the rule. What is Case V? What is a quadratic trinomial? Solve an example under Case V. Give the rule. What is Case VI in factoring? When is the difference of the same powers of two quantities divisible by the difference of the quantities? Solve examples illustrating the principle. What is the order and arrangement of the quantities in the quotient? What are the signs of the terms in the quotient? What is a demonstration? Demonstrate the principle just stated. When is the difference of the same powers of two quantities divisible by the sum of the quantities? State the principle and demonstrate it. What are the signs of the terms in the quotient? What is Case VII? When is the sum of the same powers of two quantities divisible by the sum of the quantities? State the principle and demonstrate it. When is the sum of the same powers of two quantities divisible by the difference of the quantities? State the principle and demonstrate it. Write out the quotient of $(x^5+y^5)\div(x+y)$; $(x^5-y^5)\div(x-y)$; $(x^4-y^4)\div(x+y)$.

What is a common divisor of two or more quantities? What is the greatest common divisor? What would be a more appropriate term to apply to literal quantities? Why? When are quantities prime to each other? What is the principle relating to the great-

est common divisor? What is Case I? Solve an example. Give the rule. What is Case II in greatest common divisors? Give the principles included under Case II. Show the truth of these principles by examples. Solve an example under this case and give an explanation of the process. Give the rule. What changes may be made upon the quantities whose greatest common divisor is sought without affecting the greatest common divisor?

What is a multiple of two or more quantities? What is a common multiple? What is the least common multiple? What would be a more appropriate term to apply to literal quantities? What is the principle relating to least common multiple? Give the rule for finding the least common multiple. Solve an example.

What is a fraction? What is the unit of a fraction? What is a fractional unit? How many quantities are required to express a fraction? Why? What is the denominator of a fraction? What is the numerator? What are the terms of a fraction? What are fractional forms? Define an entire quantity; a mixed quantity. What is the sign of a fraction? To what does it belong? Illustrate its use by an example.

What is meant by reduction of fractions? What is Case I in reduction? When is a fraction in its lowest terms? What principle applies to the reduction of fractions to higher or lower terms? Give the rules. What is Case II? Solve an example and give the rule. What must be done, in examples under Case II, when the sign of the fraction is —? What is Case III? Solve and explain an example. Give the rule. What is Case IV? What is the principle? Solve an example. Give the rule. What is Case V? What are similar fractions? Dissimilar fractions? When have fractions their least common denominator? State the principles relating to the common and least common denominators of fractions. Solve an example. Give the rule. What should be done with mixed quantities before finding their least common denominator? What is meant by clearing an equation of fractions? What is the principle? Upon what axiom is it based? Solve an example, explain it, and deduce the rule. What must be done, in clearing an equation of fractions, if a fraction has the minus sign before it? What effect upon a fraction has multiplying it by its denominator? Solve an equation containing fractions.

What is the principle relating to addition of fractions? Give the

rule. What is the principle relating to subtraction of fractions? Give the rule. What is the principle relating to multiplication of fractions? What is Case I? Solve an example. Give the rule. What is Case II? Solve an example. Give an explanation of the process. Give the rule. What should be done to shorten the process when possible? What is Case I in division of fractions? Solve an example. Give the rule. What is Case II? Solve an example. Give the rule. How should entire and mixed quantities be treated before dividing? What should be done, when possible, to shorten the process? Solve an example. Give the rule. What are complex fractional forms? How are they simplified?

Define an equation; members of an equation; first member; second member; clearing of fractions; transposing an axiom; a statement of a problem; a solution of a problem. Give the axioms. How is the degree of an equation determined? Write equations of the first, of the second, and of the third degrees. What is an equation of the first degree called? Of the second? Of the third? What is a numerical equation? A literal equation? Illustrate each by examples. When the same expression is found in several terms of an equation, how may the solution be shortened? Illustrate. What are the directions for solving a problem? How may fractions be avoided in the solution of problems? How may problems, in which the ratio of the numbers is given, be solved? Illustrate. What is a general problem? By assigning numerical values to the literal quantities, how many results can be obtained?

What are simultaneous equations? What are derived equations? What are independent equations? What are indeterminate equations? What are the principles relating to indeterminate equations and simultaneous equations, containing two unknown quantities? What is elimination? What is Case I in elimination? What is the principle? Solve an example. Give the rule. What is Case II in elimination? Solve an example. Give the rule. What is Case III in elimination? Solve an example. Deduce the rule. When there are three or more unknown quantities, how many independent equations must there be? Give the principle. Solve an example containing three or more unknown quantities, and deduce the rule. Give some expedients that may be resorted to in the solution of equations containing several unknown quantities.

What is Principle 1 relating to zero and infinity? Prove it. What is Principle 2? Prove it. What is Principle 3? Prove it. What is Principle 4? Prove it. What is Principle 5? Prove it. Express, by algebraic formulas, the five principles just given. Solve and interpret problems involving the principles of zero and infinity. Solve a general problem and derive a general rule from the results.

What is involution? A power? An exponent? How are powers named? What are the principles relating to the signs of the powers of positive and of negative quantities? What is Case I in involution? Give the rule. How is a fraction raised to any power? What is Case II? What is Case III? Give the principle relating to the square of a polynomial. What is Case IV? Give the principles relating to the binomial theorem. Solve an example illustrating the application of the principles.

What is evolution? What is a root? How are roots named? What is the radical, or root, sign? What is the index of a root? What is the index when none is expressed? For what are fractional exponents used? What does the numerator of a fractional exponent indicate? What the denominator? Why? What are the principles relating to the signs of roots? What is Case I in evolution? Give the rule. What is Case II? Give an explanation of the solution of an example and deduce the rule. How is the root of a fraction found? What is Case III in evolution? Solve an example under this case and deduce the general rule for the extraction of the square root. Give the principles relating to the figures required to express the square of a number and the orders in the square root of a number. What is the principle relating to the square of a number composed of tens and units? Extract the square root of a number, explain the process, and deduce the rule. What is Case IV? Solve an example under this case, explain the process, and deduce a rule from the solution. Show how the formula for obtaining the complete divisor in extracting any root of a quantity may be obtained. What are the principles relating to the number of figures required to express the cube of any number and the orders in the cube root of a number? What is the principle relating to the cube of any number composed of tens and units? Solve an example in cube root, explain the process, and give the rule. How are decimals pointed off into periods? How may a rule for the extraction of any root be formed?

What is a radical quantity? How may the root be indicated? Illustrate. What is the coëfficient of a radical? How is the degree of a radical determined? What are similar radicals? What is a rational quantity? What is a surd or irrational quantity? Illustrate. What is the principle relating to the root of the factors of a quantity? What is Case I in reduction of radicals? When is a radical in its simplest form? Solve an example under Case I, and give the rule. When is a fractional radical in its simplest form? Solve an example illustrating the reduction of a fractional radical to its simplest form, and give the rule. What is Case II? Give the rule. How may the coëfficient of a radical be placed under the radical sign? What is Case III? Give the rule. What is the principle relating to addition and subtraction of radicals? Give the rule for addition; for subtraction; for multiplication; for division. Give the rule for the involution of radicals. Solve an example under evolution of radicals, and give the rule. What is meant by rationalization? What is Case I in rationalization? Solve an example and give the rule. What is Case II? Solve an example, explain the process, and give the rule. What is Case III? Give the rule. What is an imaginary quantity? Illustrate. Give the principle relating to the form of imaginary quantities. How are imaginary quantities added and subtracted? How are imaginary quantities multiplied? What is the principle relating to the sign of the product of two imaginary quantities? Show that it is correct.

What is a radical equation? Give the suggestions to guide in the solution of radical equations.

What is a quadratic equation? A pure quadratic equation? Illustrate. By what other name is a pure quadratic equation sometimes known? What is a root of an equation? What is the principle relating to the roots of a pure quadratic? Solve an example to illustrate the truth of the principle.

What is an affected quadratic equation? Illustrate it. By what other name is an affected quadratic equation sometimes known? What is the principle relating to the roots of an affected quadratic? To what general form may affected quadratics be reduced? What is the first rule for the completion of the square? Solve an example by this rule and give the reason for the steps. Give the rule for writing the value of the unknown quantity in an affected quadratic. If the

sign of the second power of the unknown quantity is negative, what must be done before finding the value of the unknown quantity? When may the Hindoo method of completing the square be employed? Explain the process. Give the rule. When the coëfficient of the unknown quantity is an even number, how may the square be completed? Explain the process. How may the square be completed when the coëfficient of the highest power is a perfect square? Solve an example and explain the process.

When is an equation in the quadratic form? What is the general form for quadratic equations? Solve an equation in the quadratic form having fractional exponents. Solve an equation in the quadratic form in which the terms are polynomials. What is meant by the absolute term? Solve a general quadratic equation, and from the solution deduce the principles relating to the formation of quadratic equations.

What is a homogeneous equation? Into what classes may simultaneous quadratic equations, which can be solved by the rules for quadratics, be grouped? Solve an example illustrative of each class.

Define ratio; geometrical ratio; arithmetical ratio. When should the first term of a ratio be regarded as the dividend? When may either term be regarded as the dividend? What are the terms of a ratio? Define the antecedent; the consequent. What is the sign of ratio? What is a couplet? What is a simple ratio? How are ratios compounded? What is a duplicate ratio; a triplicate ratio? Illustrate each. Give the principle relating to the changes that may be made upon a ratio without changing the ratio of the terms.

What is a proportion? What is the sign of proportion? Define the antecedents of a proportion; the consequents; the extremes; the means; a mean proportional. Upon what are the changes that may be made upon a proportion based? What is principle (1) in proportion? Demonstrate it. Illustrate the truth of the principle with numbers. What is principle (2)? Demonstrate it. Illustrate its truth with numbers. What is principle (3)? Demonstrate it. What is principle (4)? Demonstrate and illustrate with numbers. What is principle (5)? Demonstrate and illustrate with numbers. What is principle (6)? Demonstrate and illustrate with numbers. What is principle (7)? Demonstrate and illustrate with numbers. What is 'nle (8)? Demonstrate and illustrate with numbers. What is

principle (9)? Demonstrate and illustrate with numbers. What is principle (10)? Demonstrate and illustrate with numbers. What is principle (11)? Demonstrate and illustrate with numbers. What is principle (12)? Demonstrate and illustrate with numbers. Solve a problem illustrating the application of the principles of proportion. Show how certain fractional equations may be solved by proportion.

What is a series? What are the extremes of a series? What are the means? What is an ascending series? What is a descending series? What is an arithmetical progression? What is the common difference? What is Case I? Give the fundamental formula for finding the last term. Show how it is deduced. What is Case II? Give the fundamental formula for finding the sum. Show how it is deduced. How may the formulas for finding any element be obtained? Give the various ways of representing the unknown terms in an arithmetical progression. What is a geometrical progression? What is the ratio? What is Case I? Show how the fundamental formula for finding the last term is obtained. What is Case II? Show how the fundamental formula for finding the sum of a series may be deduced. How may the formulas for finding any element be obtained? How may the unknown terms in a geometrical series be represented sometimes?

What is the logarithm of a number? What is a base of logarithms? What is meant by the common system of logarithms? What is meant by the characteristic of a logarithm? The mantissa? What are the principles relating to the characteristics of logarithms? Explain the construction of the tables of logarithms. How may the logarithm of a number be found? How may a number be found whose logarithm is given? How may numbers be multiplied by the use of logarithms? How may numbers be divided by the use of logarithms? How may numbers be raised to any power by logarithms? How may the roots of numbers be extracted by the use of logarithms?

ANSWERS.

Page 8.

2. Coat, \$24; vest, \$6.
3. Henry, \$9; James, \$27.
4. 8 bu.; 16 bu.
5. B, \$200; A, \$600.
6. 1st, 50; 2d, 100; 3d, 300.
7. Charles, 70; William, 280.
8. 130.

Page 9.

9. Cow, \$50; horse, \$200.
10. B, 105; A, 315.
11. Sister, 120; brother, 360.
12. Less, 90; greater, 450.
13. B, \$350; A, \$1400.
14. Wheat, 220 bu.; corn, 1100 bu.
15. Rye, 150 bu.; corn, 300 bu.; wheat, 900 bu.
16. A, \$80; B, \$160; C, \$320.
17. 1st, 13; 2d, 39; 3d, 117.
18. \$7260.
19. 1st yr., \$3450; 2d yr., \$6900.

Page 10.

20. 1st, \$75; 2d, \$225.
21. A owns 2000; B, 6000; C, 2000.
22. 3.
23. 2 ducks.
24. Younger daughter, \$1000; elder, \$2000; son, \$3000.
25. 15 slate-pencils.
26. 1st part, 2; 2d, 16; 3d, 6; 4th, 12.
27. 15.
28. 4.
29. 13.
30. B, \$3100; A, \$12400.
31. Daughter, \$1200; son, \$3600; widow, \$9600.
32. Barley, 4; oats, 12; wheat, 16.

Page 11.

33. 1st, 12; 2d, 36; 3d, 96.
34. Cherry, 20; peach, 60; apple, 480.
35. John, 6 cts.; James, 36 cts.
36. Fiction, 4500.
37. Sarah, 10 cts.; Mary, 50 cts.
38. 1st, 62; 2d, 124; 3d, 31.
39. 1st yr., \$1000; 4th yr., \$8000.
40. A, \$3000; B, \$2000; C, \$1000.

Page 17.

1. 5.
2. 4.
3. 11.
4. 15.
5. 5.
6. 1.
7. 1.
8. 9.
9. 3.
10. $1\frac{2}{3}$.
11. 4.
12. 70.
13. 240.
14. 36.
15. 6.
16. 9.
17. 8.
18. $2\frac{8}{11}$.
19. 5.
20. 6.
21. 12.
22. 13.
23. $27\frac{9}{16}$.
24. $33\frac{1}{5}$.

Page 21.

3. $26b$.
4. $19ax$.
5. $25x^2y$.
6. $-23z^2y^2$.
7. $-13cx^3$.
8. $30ax$.
9. $26mn$.
10. $-26x^2y^2$.
11. $19x^3y^3$.
12. $4a$.
13. a^3x.
14. $6\sqrt{xy}$.
15. $-4(xy)^3$.
16. $(x+y)^4$.

Page 22.

3. $6a+4b-6c$.
4. $7x-6xy+z$.
5. $8x+9z-2xz$.

Page 23.

6. $-2y+6z$.
7. $2xy+6z-6y+4x$.
8. $4ac+4ay$.
9. $16b+5cd-13e$.
10. $-3x^2y+6xy+z$.
11. $-3a+8c+8d$.
12. $-5y+5w+z$.
13. $10a^2b^2-7c^3y^3+d^2$.
14. $7ab+9\sqrt{xy}+25$.
15. $12x^3+3x+6$.
16. $5\frac{13}{16}ax^2+5\frac{2}{3}a^2+5\frac{3}{8}x^3y-2\frac{1}{15}b^3$.
17. $2ab^2+\frac{1}{4}a^3+3\frac{3}{4}abc+1\frac{1}{3}a^2c+b^3+1\frac{1}{2}b^2c+c^3$.
18. $8(x+y)$.
19. $7(a-b)^2+9(x-y)^2$.

Page 24.

21. $(2a-3b+4c+3d)x$.
22. $(2a-4b+3c+4)x^2$.
23. $(6+5a)(a+b)$.
24. $(3a+2b+7)(a+3)$.
25. $(5a+5)\sqrt{x+y}$.

2. 2.	5. 4.	8. 9.
3. 4.	6. 4.	9. 2.
4. 2.	7. 4.	10. 4.

11. Harvey, 7; Henry, 21; James, 42.

Page 25.

12. C, \$20; B, \$40; A, \$80.
13. Samuel, 5; Henry, 15; William, 30.
14. B, \$200; C, \$400.
15. 10.
16. Fiction, 2000; reference, 20000; historical, 6000.
17. A, \$20000; B, \$2000; C, \$6000; D, \$8000.
18. A, 612; B, 306; C, 204.
19. Board, \$36; wages, \$60.

Page 28.

3. $9a$.
4. $5xy$.
5. $-2x^3y^2$.
6. $-3xyz$.
7. $-12x^2y^3z$.
8. $-3a^2b^3c$.

Page 29.

9. $4x+4y$.
10. $a-b$.
11. $-2xy+2z$.
12. $2x^2y^2-3z$.
13. $-2xy^3z-xy$.
14. $p^2qs+3pq^2s$.
15. $2m^2nx-mnx$.
16. $4x^2y+5y^2$.
17. $-2xy^2-3z$.
18. $-4p^2q^2-pq$.
19. $14x^2y^2z^2-2y^2$.
20. $-5yz^4+2y^4z$.
21. $6p^2q^2-2qs$.
22. $-9xyz^3-3xyz$.
23. $-4a^2xy+2ax^2$.
24. $r^2s^2z-4rsz^2$.

Page 30.

3. $6a^3x$.
4. $8x^3y^3$.
5. $4x+3y$.
6. $3y-6z$.
7. $4ax+5by$.
8. $a+6b-8c$.
9. $2x+7y+2z$.
10. $2xy+6z-3x^2+y$.
11. $-b+2c$.
12. $x+5y-7z$.
13. $3a^2+5b^2+5c^2$.
14. $6a^3-6c^3-6d^3$.
15. $11x^4-7y^2$.
16. $13p^2+6q^2-2r^3$.
17. $-ax+4ay$.

Page 31.

18. $8yz+7xz$.
19. $-6x^3y^2+24xy^3+14xy$.
20. $9x^3y^3-11yz^3$.
21. $x^2+5xy+5z^2+w$.
22. $15x^3+5y^3+4z^3-7r^3$
23. $3x^5y+7x+4$.
24. $5bx^3+3ay^2-2by+9$.
25. $x^2y^4+5xy-9x+5$.
26. $2x^5y^2-9xy^5-9z^4-9$.
27. $4ar^2-6bs^3+3rs-p-7$.
28. $15x^5-39x^3y^3-11y^4-4z-z^5$.
29. $-x^m-6x^ny^m+4y^m+4x^{2m}$.
30. $3x^{2n}-4x^{3n}y^m-4y^{m-1}+4x^{2m}$.
31. $\sqrt{xy}+5z+\sqrt[3]{y^2}$.
32. $6(a+b)^2-4a+6c$.
33. $5\sqrt{a+b^2}-9\sqrt[3]{x+y}+7\sqrt{x+y}$.
34. $\sqrt{a+b^2}-5\sqrt[3]{c+d}$.

Page 32.

36. $(a-c)y+(d+2)x$.
37. $-(a+b)x+4(c+d)y$.
38. $(c-b)x+(a+2b-c)y$.
39. $(a-b)x+(a+b)y+(c-1)z$.
40. $(5a-c)y+(a+2c)z+(d-6)x$.
41. $(a-2b)x^2+(3a+2c)y+(3+c)x^3y$.

Page 33.

1. $-b$.	6. $x+2y$.
2. y.	7. $2a-y$.
3. $2a+b$.	8. $x+4y$.
4. $2a+b$.	9. $7x-7y$.
5. b.	10. $6x-y+2z$.

11. $x-z^2$.
12. $-xy+3x^3y-x^2$.
13. $7x^2+4y^2+z^2$.
14. $6ab^2+4ac^2$.

Page 34.

15. $3a-2b+c-2d$.
16. $-5x^2+7x^3+6y$.
17. $-5x^2y+4y+1$.
18. $ab-2bc-4bd-6c$.
19. $6xy+11z$.

Page 37.

2. 4.	8. 2.	14. 4.
3. 8.	9. 5.	15. 5.
4. 12.	10. 3.	16. $1\frac{1}{2}$.
5. 7.	11. 4.	17. 6.
6. 10.	12. 2.	18. 4.
7. 4.	13. 3.	19. 7.

Page 38.

20. 8.	25. 8.	31. 18.
21. 3.	27. 46.	32. 10.
22. 11.	28. 18.	33. 12.
23. 2.	29. 43.	34. 11.
24. 17.	30. 15.	35. 13.

Page 39.

37. John, 20; James, 30; Henry, 35.
38. In 1st, 110; 2d, 130; 3d, 155.
39. 48.
40. \$1500; \$1650; \$1800; \$1950.
41. \$2000; \$2250; \$2500; \$2750.

Page 44.

3. -24.
4. -12.
5. $21a$.
6. $-12x$.
7. $-20x$.
8. $6x^6$.
9. $6x^7$.
10. $8x^5y^2$.
11. $6x^5y^6$.
12. $-12x^3m^2y^3$.
13. $-40x^3y^5z^3$.
14. $16x^4y^3z^3$.
15. $-24a^3b^3x^4$.
16. $15a^6b^4x^2y^2$.
17. $-12c^4d^2y$.
18. $-15a^4x^6y^3z$.
19. $24x^4y^4z^3$.
20. $-12a^2x^2y^4z^2$.
21. $-15abx^4yz$.
22. $2(x+y)$.
23. $-12(a+b)$.
24. $15(y+z)^5$.
25. $4(a-b)^6$.
26. $6(c+d)^5$.

Page 45.

27. $-10(x+y+z)^7$.
28. $12x^{2n}$.
29. $-20a^{3n}$.
30. $-15a^3x^{3n+1}$.
31. $8a^2x^{m+n}$.
32. $-15a^{3n}x^{4n}$.
33. $20x^{m+n}y^{m+n}$.
34. $3x^2y - 6y^2$.
35. $2x^2yz - 4z^2$.
36. $12x^3y - 6x^2y^2$.
37. $-6x^4y - 4x^2y^3$.
38. $-16x^4y^2z^2 - 8x^2z^4$.
39. $9x^3y^3z - 6xy^2z^2$.
40. $4x^4y + 2xy^2 + 3xyz$.
41. $6x^3yz + 2xyz - 6x^2z^2$.
42. $18x^3y^4 + 12xy^4 - 18xy^2z^2$.
43. $12a^2bcd - 9a^2c^2d - 9a^2cd^2$.
44. $-25a^2c^2x + 30a^2cx^2 - 20a^2bcx$.
45. $-20a^2b^2c^2d + 12a^2bc^2d^2 + 12ab^2c^2d^2$.
46. $-6a^3x^3y + 4a^3bcx^2 - 3a^2x^3y$.

Page 46.

4. $x^2 - y^2$.
5. $3a^2 + 10ac + 3c^2$.
6. $12a^2 - 18ab + 6b^2$.
7. $6y^2 + yz - 12z^2$.
8. $4x^2 + 6xy + 2y^2$.
9. $9x^2 - 24xy + 16y^2$.
10. $15a^2 - 29ac - 14c^2$.
11. $a^2x^2 + 2abxy + b^2y^2$.
12. $4a^2c^2 - 9b^2c^2$.
13. $6b^2d^2 + b^2cd - 12b^2c^2$.
14. $6x^4y^4 + x^2y^2z^2 - 12z^4$.
15. $6x^3y^2z^2 + 4xy^3 + 3x^2z^3 + 2yz$.
16. $8a^2b^3 + 6ab^2c^2 + 8ab^3c^2 + 6b^2c^4$.
17. $25x^4y^4 - 15ax^2y^3 - 10ax^4y + 6a^2x^2$.
18. $a^3 + 3a^2b + 3ab^2 + b^3$.
19. $x^3 + 6x^2 + 12x + 8$.
20. $a^3 - 2ay^2 + y^3$.
21. $6a^3 - 3a^2b - 9ab^2 + 6b^3$.
22. $a^3 - a^2$.

Page 47.

23. $x^6 - y^6$.
24. $6x^2 - 5xy + 2xz - 6y^2 + 23yz - 20z^2$.
25. $6a^4 + 11a^3b + a^2c^2 - 10a^2b^2 + 31abc^2 - 15c^4$.
26. $21x^4 - 34x^3y + 34x^2y^2 + 2xy^3 - 15y^4$.

27. $1 - 5x + 11x^2 - 12x^3 + 6x^4$.
28. $a^4 + a^2x^2 + x^4$.
29. $x^2 + 3xy - 3xz + 2y^2 - 5yz + 2z^2$.
30. $a^{2m} - b^{2m}$.
31. $x^{2n} + 2x^ny^n + y^{2n}$.
32. $x^{m+n} + x^ny^m + x^my^n + y^{m+n}$.
33. $x^{2m+2n} + 2x^{m+n}y^{m+n} + y^{2m+2n}$.
34. $a^{2m-2n} - b^{2m-2n}$.
35. $x^2 + 2xy + y^2$.
36. $4x^2 - 4xy + y^2$.
37. $9x^2 - 16y^2$.
38. $16x^2 - 36y^2$.
39. $9a^2x^2 + 6axy + 6axz + 4yz$.
40. $4x^2 - 8x^2y - 4xz + 8xyz$.
41. $9a^3 - 6abc + 6a^2bc - 4b^2c^2$.
42. $a^3 + ab + a^2b^2 + b^3$.
43. $a^2 - b^2 - 2bc - c^2$.
44. $a^3 + 3a^2b + 3ab^2 + b^3$.
45. $a^4 - 2a^2b^2 + b^4$.
46. $x^4 - 2x^2 + 1$.
47. $a^4 - 2a^2b^2 + b^4$.
48. $1 + 2a - 2a^3 - a^4$.
49. $x^8 - 4x^6y^2 + 6x^4y^4 - 4x^2y^6 + y^8$.
50. $a^{16} - 2a^{12}b^4 + 2a^4b^{12} - b^{16}$.
51. $a^{16}b^{16} - 2a^{12}b^{16} + 2a^4b^{16} - b^{16}$.

Page 48.

2. 6.
3. 4.
4. 2.
5. 17.
6. $1\frac{7}{9}$.
7. 2.
8. 12.
9. -1.
10. 8.
11. $2\frac{1}{2}$.

Page 49.

12. 38.
13. 4.
14. 10.
15. 14.
16. 25.
17. $1\frac{1}{2}$.
18. 6.
19. 3.
20. 1.
21. 17.
23. 4.
24. A, 10; B, 10.
25. Henry, 9; John, 12.
26. 1st, 20; 2d, 35.

Page 50.

27. C, \$5; B, \$10; A, \$20.
28. Smaller, 10; larger, 40.
29. B, \$1400; A, \$2800.
30. Amount wanted, 46 lbs.; in 1st firkin, 40 lbs.; in 2d firkin, 60 lbs.

Page 51.

1. $c^2 + 2cd + d^2$.
2. $m^2 + 2mn + n^2$.
3. $r^2 + 2rs + s^2$.
4. $x^2 + 4x + 4$.
5. $a^2 + 6a + 9$.
6. $9a^2 + 6ax + x^2$.
7. $4x^2 + 16xy + 16y^2$.
8. $9a^2 + 12ab + 4b^2$.
9. $x^4 + 2x^2y^2 + y^4$.
10. $16x^2 + 24xy + 9y^2$.
11. $9p^2 + 12pq + 4q^2$.
12. $4x^4 + 20x^2y^2 + 25y^4$.

Page 52.

13. $a^2 - 2ac + c^2$.
14. $y^2 - 2yz + z^2$.
15. $r^2 - 2rs + s^2$.
16. $b^2 - 2bc + c^2$.
17. $x^2 - 2x + 1$.
18. $x^2 - 4xy + 4y^2$.
19. $a^2 - 2ad + d^2$.
20. $4r^2 - 12rs + 9s^2$.
21. $4s^2 - 4sq + q^2$.
22. $9m^2 - 24mn + 16n^2$.
23. $4v^2 - 4vw + w^2$.
24. $4x^4 - 8x^2y^2 + 4y^4$.
25. $c^2 - d^2$.
26. $r^2 - s^2$.
27. $m^2 - n^2$.
28. $c^2 - a^2$.

29. $x^2-1.$
30. $4-x^2.$
31. $4x^2-16.$
32. $4x^4-y^2.$
33. $x^4-y^4.$
34. $x^8-y^8.$
35. $9v^2-4w^2.$
36. $25x^2y^2-9.$

Page 53.

37. $x^2+7x+12.$
38. $x^2-2x-15.$
39. $x^2-x-12.$
40. $x^2-10x+24.$
41. $a^2+(3+b)a+3b.$
42. $a^2+(m+n)a+mn.$
43. $4x^2-2x-20.$
44. $9x^2-6x-35.$
45. $4y^2-14y+12.$
46. $16a^2+(b+c)4a+bc.$
47. $25a^2+(2b-2c)5a-4bc.$
48. $9a^2x^2-9ax-28.$
49. $4a^4x^2-8a^2x-12.$
50. $4x^4y^6+22x^2y^3+28.$

Page 57.

4. $2.$
5. $-4.$
6. $-3ay.$
7. $4xy.$
8. $-3z^2.$
9. $4xyz.$

Page 58.

10. $-5xyz.$
11. $2a^4b^4.$
12. $2f.$
13. $2x^2.$
14. $-2xz.$
15. $-3w.$
16. $-3rz^2.$
17. $7n.$
18. $2y^2.$
19. $-2ax^2y^2.$
20. $4s.$
21. $-18x.$
22. $-24.$
23. $-9n^ny^n.$
24. $-5x^{-1}y^{-1}z.$
25. $\frac{-7z^{2m}}{z^3x}.$
26. $\frac{-5n^2}{m^2}.$
27. $x+y.$
28. $a(x+y).$
29. $-3(x+z).$
30. $20.$
31. $18(x+z)^3.$
32. $a(x-y).$
33. $-\frac{5a}{2}(c+d)^3.$
34. $-2x^2.$
35. $ax-2y.$
36. $3y-3x.$
37. $2x+y.$
38. $ab-2b^2.$
39. $ab-c.$
40. $3xy+z.$
41. $-3xy+ax.$
42. $-6xyz+4.$
43. $a-3b+c^2.$
44. $x-y+xy^2.$
45. $x-2y+\frac{y^2}{x}.$
46. $z-3x+3z.$
47. $m+2-\frac{3m^2}{mn}.$
48. $c-3d+\frac{4d^3}{cd}.$
49. $1+3x-\frac{3a^2y}{a^2x}.$

Page 59.

50. $v+3vy-\frac{2y^2}{vy}.$
51. $-3+2(x+y).$
52. $-a(b+c)-b(b+c)^2.$
53. $3-2(a-c)^2.$
54. $-(x+z)^2+2(x+z)^3.$

Page 61.

6. $a-b.$
7. $x+2.$
8. $3+x.$
9. $x^2+y^2.$
10. $a^3+y^3.$
11. $x^2+2xy+y^2.$
12. $r+s.$
13. $x^3+3x^2y+3xy^2+y^3.$
14. $c^2+2cd+d^2.$
15. $x^2+5x+7.$

Page 62.

16. $a + x$.
17. $a + b - c$.
18. $a^2 - 2ay + y^2$.
19. $x^2 - ax - b$.
20. $5a^2 + 2ab - 3b^2$.
21. $3x^2 - 5y^2 + 3z^2$.
22. $2a^2 - 3a + 1$.
23. $2a + 3b$.
24. $6b^2 + 12ab + 27a^2 - 1$.
25. $5a^3 + 4a^2 + 3a + 2$.
26. $x^2 - xy + y^2 - xz - yz + z^2$.
27. $6x^2 - 7x + 8$.
28. $8x^3 + 12ax^2 - 18a^2x - 27a^3$.
29. $a^2 - 2ax + 4x^2$.
30. $x^2 + xz + z^2$.
31. $x^3 + x^2y + xy^2 + y^3$.
32. $x^4 - x^3y + x^2y^2 - xy^3 + y^4$.
33. $x^6 - x^5 + x^4 - x^3 + x^2 - x + 1$.
34. $x^3 + 3x^2y + 9xy^2 + 27y^3$.
35. $27a^3 - 18a^2b + 12ab^2 - 8b^3$.
36. $x^{n-1} - x^{n-2}y + x^{n-3}y^2$, etc.

Page 63.

1. $-a^0b^0x^0$, or -1.
2. $2a^0x$, or $2x$.
3. $-2a^2x^0$, or $-2a^2$.
4. $-3xy^0z$, or $-3xz$.
5. $6x^3(y + z)^0$, or $6x^3$.
6. $\frac{1}{x^{-2}}$, or x^2. 7. $\frac{1}{x^{-2}y^{-2}}$, or x^2y^2.
8. $\frac{1}{a^2x^{-2}y^{-3}}$, or $\frac{x^2y^3}{a^2}$.
9. $12x^2y^{-11}$.
10. $-4a^{-2}b^{-3}c^2$.
11. $\frac{x^3y^2}{2}$. 12. $\frac{x^my^n}{3}$.

Page 64.

3. $c + 3$.
4. $a - 4$.
5. $2a + 3$.
6. $d - 3a$.
7. $a + b$.
8. $2a + 3b$.

Page 65.

9. $a^2 + c$.
10. $3a + 2b$.
11. $7 + 5b$.
12. $5a + 2b$.
13. $2c^2 - d$.
14. $b - 3c$.
15. $2m^2 - 3m + 1$.
16. $9 - 6a + a^2$.
17. $2m^2 + 3mn + n^2$.
18. $3a^2 - b + b^2$.
19. \$110.
20. 1st, \$1000; 2d, \$2000; 3d, \$4000; 4th, \$3500.
21. 5 beggars; 19 cents.
22. 6th, 10 years; 5th, 14 years; 4th, 18 years; 3d, 22 years; 2d, 26 years; 1st, 30 years.
23. 7 gallons.
24. \$$\frac{7}{8}$.

Page 66.

1. $20ax + 20 + 8\sqrt{x} + 10x^2$.
2. $2am + 5x + 3\sqrt{y} + z - x^2$.
3. $4\sqrt{a^2 - b^2} - 6(x + y) - 6$.
4. $3\sqrt{x} - \sqrt{y} - 3z + 22 + y$.
5. $(a^2 - b^2)x^2 - ay + (c - 3)y^5 - 3z^6$.
6. $-4x^{2n} - 6x^2y^2 + 6x^2y^n - 6yz + 4az + 4x^m + 6z$.
7. $x^6 + 2x^4y + x^3y^6 + 2x^5y + 4x^3y^2 + 2x^2y^4 - x^4y^2 - 2x^2y^3 - xy^5$.
8. $x^{2n} + 4x^{2n}y^n + 2x^ny^n + 4x^{2n}y^{2n} + 4x^ny^{2n} + y^{2n}$.
9. $3 + 2x^{-n}y^{-2n} - x^ny^n - 3x^{-n}y^{2n} - 2x^{-2n} + y^{3n} + 3y^n + 2x^{-n}y^{-n} - x^ny^{2n}$.

10. $9x^n + 6x^{-2}y^{n+m} + 3x^{-2}z^m -$
$6x^{n+2}y^{-n-m} - 4 - 2y^{-n-m}z^m +$
$3x^{n+2}z^{2m} + 2y^{n+m}z^{2m} + z^{3m}$.
11. $x^5 + 5x^4y + 10x^3y^2 +$
$10x^2y^3 + 5xy^4 + y^5$.
12. $a^4 - 8a^2 + 16$.
13. $81a^4 - 648a^2 + 1296$.
14. $x^2 + 4xy + 4y^2$.
15. $4x^2 + 20xy + 25y^2$.
16. $9x^4 - 12x^2y^2 + 4y^4$.
17. $x^{4n} + 4x^{2n}y^{2n} + 4y^{4n}$.
18. $x^{-4n} - 4x^{-2n}y^{-2n} + 4y^{-4n}$.
19. $4x^2 - y^2$.
20. $9x^2 - 49y^2$.
21. $16x^4 - 4y^4$.
22. $a^2x^{2n} - y^{2n}$.
23. $a^2x^{-2n} - a^2y^{-2n}$.

Page 67.

24. $a^8 - x^8$.
25. $x^{16} - y^{16}$.
26. $256x^8 - 2592x^4 + 6561$.
27. $2a^2 + 3ab + b^2$.
28. $x + y$.
29. $x^3 - 3x^2y + 3xy^2 - y^3$.
30. $2a^{2n} - 4a^nb^n + 2b^{2n}$.
31. $\frac{1}{x^5y^3z^4}$.
32. $\frac{x^2}{y^3z^4}$.
33. $\frac{y^2z^2}{a^2}$.
34. $\frac{r^5}{s^4z^5}$.
35. b^{-2}.
36. $a^{-1}xy^{-1}$.
37. $-x^nz^{-m}$.
38. $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$.
39. $x^{-4} + 4x^{-3}y^{-1} + 6x^{-2}y^{-2} +$
$4x^{-1}y^{-3} + y^{-4}$.
40. $2a - 3b$.
41. $3 + 2a + c$.
42. $2a + 3c + d$.

Page 69.

2. 2, 2, 2, a, a, b.
3. 2, 5, x, x, y, y, y.
4. 3, 5, a, a, a, y, y, s.
5. 2, 2, 5, a, x, x, x, y.
6. 2, 3, 7, a, x, y, y, y.
7. 2, 2, 3, 3, x, y, y, z, z, z.
8. 2, 2, 7, a, a, c, c, x.
9. 5, 7, c, c, c, x, x, z, z.

Page 70.

2. $a^2(5b + 6c)$.
3. $4x^2(2y^2 + 3z^2)$.
4. $6xyz(1 + 2xy)$.
5. $9xy^2z(x^2 + 2z^2)$.
6. $a^2xyz(xy + z)$.
7. $c(a^2 + b^2 + cd^2)$.
8. $xy(4x + cy + 3y^2)$.
9. $a^2z(ay + x + x^2y^2z)$.
10. $bxy^2(bx + b^2 + xyz)$.
11. $ax^nyz(ay^{n-1} + z^{n-1} + ayz)$.

Page 71.

2. $(a + b)(a + b)$.
3. $(x + y)(x + y)$.
4. $(b - c)(b - c)$.
5. $(r + s)(r + s)$.
6. $(x + 1)(x + 1)$.
7. $(x + 2)(x + 2)$.
8. $(y - 1)(y - 1)$.
9. $(2y - 1)(2y - 1)$.
10. $(3x + 1)(3x + 1)$.
11. $(3m + 3n)(3m + 3n)$.
12. $(3 + x)(3 + x)$.
13. $(1 - x^2)(1 - x^2)$.
14. $(4n - 1)(4n - 1)$.
15. $(4 + 2a)(4 + 2a)$.
16. $(6 + a^2)(6 + a^2)$.
17. $(7 - x^3)(7 - x^3)$.
18. $(9r - a)(9x - a)$.
19. $(2a^n + 3b^n)(2a^n + 3b^n)$.

Page 73.

2. $(a+b)(a-b)$.
3. $(c+d)(c-d)$.
4. $(m+n)(m-n)$.
5. $(2x+2y)(2x-2y)$.
6. $(3x+y)(3x-y)$.
7. $(x+3y)(x-3y)$.
8. $(4x+4y)(4x-4y)$.
9. $(\frac{1}{2}x+\frac{1}{3}y)(\frac{1}{2}x-\frac{1}{3}y)$.
10. $(xy+2yz)(xy-2yz)$.
11. $(m^2+n^2)(m+n)(m-n)$.
12. $(a^4+b^4)(a^2+b^2)(a+b)(a-b)$.
13. $(m^n+n^m)(m^n-n^m)$.
14. $(3a^n+2b^{2n})(3a^n-2b^{2n})$.
15. $(a^8+b^4)(a^4+b^2)(a^2+b)(a^2-b)$.

Page 74.

2. $(x+2)(x+1)$.
3. $(x+4)(x+3)$.
4. $(x-7)(x+3)$.
5. $(x-9)(x+2)$.
6. $(x+4)(x+2)$.
7. $(x+8)(x+4)$.
8. $(x-13)(x+3)$.
9. $(x-16)(x+4)$.
10. $(2x-3)(2x-2)$.
11. $(3x-6)(3x-3)$.
12. $(2x+6a)(2x+2a)$.
13. $(3a+6b)(3a+4b)$.

Page 75.

1. $x^7+x^6y+x^5y^2+x^4y^3+x^3y^4+x^2y^5+xy^6+y^7$.
2. $x^8+x^7y+x^6y^2+x^5y^3+x^4y^4+x^3y^5+x^2y^6+xy^7+y^8$.
3. x^3+x^2+x+1.
4. x^3+2x^2+4x+8.
5. $x^6+x^4y^2+x^2y^4+y^6$.

Page 77.

1. $x^7-x^6y+x^5y^2-x^4y^3+x^3y^4-x^2y^5+xy^6-y^7$.
2. $x^9-x^8y+x^7y^2-x^6y^3+x^5y^4-x^4y^5+x^3y^6-x^2y^7+xy^8-y^9$.
3. x^3-x^2+x-1.
4. x^3-2x^2+4x-8.
5. $x^6-x^4y^2+x^2y^4-y^6$.

Page 78.

1. $x^6-x^5y+x^4y^2-x^3y^3+x^2y^4-xy^5+y^6$.
2. $x^8-x^7y+x^6y^2-x^5y^3+x^4y^4-x^3y^5+x^2y^6-xy^7+y^8$.
3. $x^4-x^3+x^2-x+1$.

Page 79.

1. $x+y+\dfrac{2y^2}{x-y}$.
2. $x^2+xy+y^2+\dfrac{2y^3}{x-y}$.
3. $x^3+x^2y+xy^2+y^3+\dfrac{2y^4}{x-y}$.
4. $x^4+x^3y+x^2y^2+xy^3+y^4+\dfrac{2y^5}{x-y}$.

Page 82.

3. $6m^2nx^2$.
4. $4r^2s^2x^2$.
5. $7x^2y^2z^3$.
6. $5x^5yz^2$.
7. axy.
8. a^2xy^2.
9. $2b^2c$.
10. $2xy^2$.
11. $2r^2s^2t$.
12. $5a^2xy^2$.
13. $3x^2y^2z^2$.
14. $a-b$.
15. $x-2$.
16. $4x-y$.
17. $x+3$.
18. $x+5$.
19. $x+6$.
20. $x+3$.
21. $x+7$.
22. $x+6$.
23. $x+y$.

Page 86.

3. $x-7$.
4. $x+4$.
5. $x-3$.
6. $x+6$.
7. $3x-2$.
8. $x-4y$.
9. $x-y$.
10. x^2-2x+1.
11. $2x^2+4x+2$.
12. $3x+9$.
13. $5x^2-1$.
14. $x+3$.
15. $x-7$.
16. $x+3$.
17. a^2-b^2.
18. $x+2$.
19. $a+b$.
20. $3x^2+2x+1$.
21. x^2+4.
22. a^2+4.

Page 89.

3. $40a^2b^2c^3$.
4. $100x^3y^3z^3$.
5. $70a^2b^2c^2x^2y$.
6. $72m^2n^3y^3$.
7. $36r^3s^3z^4$.
8. $x^3-x^2y-xy^2+y^3$.
9. $x^3+x^2y-xy^2-y^3$.
10. $x^4-2x^2y^2+y^4$.
11. $x^4+x^3y-xy^3-y^4$.
12. $a^2y^2(x^2-z^2)$.
13. x^4-1.
14. $12xy^2(x^2-y^2)$.
15. $x(x^6-1)$.
16. $x(x^4+x^3-x-1)$.
17. $8(1-x^2)$.
18. $x^3+9x^2+26x+24$.
19. $a^3-4a^2-17a+60$.
20. $x^3-11x^2-4x+44$.
21. $x^4-6x^3-6x^2+70x-75$.
22. x^4-y^4.
23. $x^5-xy^4+x^4y-y^5$.
24. $a^5+2a^4-16a-32$.
25. x^5-xy^4.
26. $x^3-3x^2-4x+12$.
27. $x^4-2ax^3+a^2x^2-10x^3+20ax^2-10a^2x+25x^2-50ax+25a^2$.
28. $x^5+2x^4-16x-32$.

Page 94.

3. $\frac{12a}{28}$.
4. $\frac{30x^2}{36}$.
5. $\frac{10a+20b}{15}$.
6. $\frac{15x+35}{30}$.
7. $\frac{6x}{18x+9}$.
8. $\frac{9x}{18x-24}$.
9. $\frac{4ax^2}{6x+4xy}$.
10. $\frac{2a^2+ax-2ab-bx}{a^2-b^2}$.
11. $\frac{3ax-ay+3bx-by}{a^2+2ab+b^2}$.
12. $\frac{x}{5}$.
13. $\frac{3}{4y}$.
14. $\frac{2}{5xy}$.
15. $\frac{2z^2}{3xym^2}$.

Page 95.

16. $\frac{7n^2}{4z}$.
17. $\frac{2z}{y}$.
18. $\frac{5x}{7yz}$.
19. $\frac{2az^2}{3y}$.
20. $\frac{a+b}{a-b}$.
21. $\frac{a-b}{a+b}$.
22. $\frac{x-1}{x+1}$.
23. $\frac{x-1}{2y}$.
24. $\frac{x^2+ax}{x-a}$.
25. $\frac{x^3}{x^2+y^2}$.
26. $\frac{x+3}{x^2-4x}$.
27. $\frac{x+4}{x-4}$.

Page 96.

2. $\frac{10x+4y}{5}$.
3. $\frac{20x-3y}{4}$.

4. $\frac{8x-6z}{2}$.

5. $\frac{4x+4y+3}{4}$.

6. $\frac{8a+3x+4}{4}$.

7. $\frac{16x+3y-4}{8}$.

8. $\frac{18x-2y-3}{6}$.

9. $\frac{10a-3x-4}{2}$.

10. $\frac{24a-3y-7}{4}$.

11. $\frac{3cd+4a+b}{d}$.

12. $\frac{4acd+3c-d}{cd}$.

13. $\frac{3ax^2+6a-x}{ax}$.

14. $\frac{5x+20+2c-d}{5}$.

15. $\frac{a^2-2ac+c^2}{a}$.

16. $\frac{x^2-9x+6}{x-2}$.

17. $\frac{2a^2}{a-x}$.

18. $\frac{a^2+2ac-2c^2}{a-c}$.

19. $\frac{2x^2-2y^2}{x+y}$.

20. $\frac{-14}{x-4}$, or $\frac{14}{4-x}$.

21. $\frac{a^2-4ax+4x^2}{a-x}$.

22. $\frac{-2ab}{a-b}$, or $\frac{2ab}{b-a}$.

23. $\frac{m^2-2mn-2n^2}{m-n}$.

Page 97.

2. $a+\frac{c^2}{a}$.

3. $x+\frac{cd}{b}$.

4. $2b-\frac{b^2}{a+b}$.

5. $a+x$.

6. $a-x$.

7. x^2-x+1.

8. $x^2+x+1+\frac{2}{x-1}$.

9. $2x$.

10. $a^2+ab+b^2+\frac{2b^3}{a-b}$.

11. $1+\frac{5ay+x}{ax}$.

12. $2a-2b$.

13. $x+y+1+\frac{y^2-y}{x+y}$.

14. a^2+ab+b^2.

15. $x+\frac{y^2}{x+y}$.

Page 99.

2. $a^4c^{-2}x^{-3}$.

3. $x^2yc^{-2}d^{-1}$.

4. x^{-1}.

5. b^2c^{-2}.

6. $x^{-1}y^{-1}z^2$.

7. $x^{-1}+a^{-1}$.

8. $yz^{-1}-c^{-1}x^{-1}$.

9. $(x-y)(x+y)^{-1}$.

10. $(a-x)^{-1}$.

11. $y^{-2}-x^{-2}$.

12. $3x^2y^2$.

13. $\frac{4a^2}{3c^2}$.

14. $\frac{3a^2x^3y}{4}$.

15. $\frac{a^4-b^4}{a^2-c^2}$.

16. $\dfrac{4}{a^3-3a^2x+3ax^2-x^3}$.

17. x^2+y^2.

18. $\dfrac{5}{x^3-3x^2-9x+27}$.

19. $\dfrac{7(x-y-z)}{5(x+y+z)}$.

Page 101.

2. $\dfrac{9x}{12}$ and $\dfrac{10x}{12}$.

3. $\dfrac{21a}{24}$ and $\dfrac{20a}{24}$.

Page 102.

4. $\dfrac{12x^2y}{16}$ and $\dfrac{2xy^2}{16}$.

5. $\dfrac{4x}{6a}$ and $\dfrac{4y}{6a}$.

6. $\dfrac{4by}{6y^2}$ and $\dfrac{2c}{6y^2}$.

7. $\dfrac{9acz}{6x^2yz}$ and $\dfrac{4bdy}{6x^2yz}$.

8. $\dfrac{4x-8y}{10x^2}$ and $\dfrac{3x^2-8xy}{10x^2}$.

9. $\dfrac{16a+20b}{12a^2}$ and $\dfrac{9a^2+12ab}{12a^2}$.

10. $\dfrac{6ax-4ay}{10a^2c}$ and $\dfrac{4x-3y}{10a^2c}$.

11. $\dfrac{18xz}{12x^2yz}, \dfrac{12az}{12x^2yz}, \dfrac{20c}{12x^2yz}$.

12. $\dfrac{2cxy^2}{8x^2y^2}, \dfrac{2dxy}{8x^2y^2}, \dfrac{d}{8x^2y^2}$.

13. $\dfrac{3cd}{3a^2c^2}, \dfrac{ac}{3a^2c^2}, \dfrac{12a^2c^2}{3a^2c^2}$.

14. $\dfrac{acx+acy}{4ac}, \dfrac{2ax-2ay}{4ac}, \dfrac{2cx^2+2cy^2}{4ac}$.

15. $\dfrac{x^2+3x+2}{x^2-1}, \dfrac{x^2-3x+2}{x^2-1}, \dfrac{x+3}{x^2-1}$.

16. $\dfrac{ax^2y-bx^2y}{a^2-b^2}, \dfrac{axy+bxy}{a^2-b^2}, \dfrac{xy^2}{a^2-b^2}$.

17. $\dfrac{x^2+2xy+y^2}{x^2-y^2}, \dfrac{x^2-2xy+y^2}{x^2-y^2}, \dfrac{x^2+y^2}{x^2-y^2}$.

18. $\dfrac{x^4-2x^2+1}{x^4-1}, \dfrac{x^4+2x^2+1}{x^4-1}, \dfrac{x^4+1}{x^4-1}$.

19. $\dfrac{a-c}{(a-b)(b-c)(a-c)}, \dfrac{b-c}{(a-b)(b-c)(a-c)}$.

Page 104.

3. 20. 4. 18. 5. 12. 6. 10.

Page 105.

7. 14.	14. 24.	21. 12.
8. 30.	15. 12.	22. 40.
9. 72.	16. $17\frac{1}{1}\frac{8}{9}$.	23. 5.
10. 5.	17. 24.	24. 8.
11. 24.	18. 48.	25. $20\frac{2}{3}$.
12. 24.	19. 7.	26. $15\frac{5}{7}$.
13. 12.	20. 1.	27. 23.

Page 106.

28. $2\frac{1}{3}$.
29. $5\frac{1}{2}$.
30. 1.
31. 11.
32. 3.
33. 2.
34. 7.
35. $-5\frac{7}{9}$.
36. 3.
37. $\frac{30}{49}$.
38. 28.
39. 6.
40. 9.

Page 107.

42. 12.
43. 18.
44. 30.

45. 1st, \$1,000; 2d, \$300; 3d, \$200.
46. B's, \$2000; A's, \$1500.
47. Horse, \$180; carriage, \$240.
48. A, $\$16\frac{5}{37}$; B, $\$8\frac{4}{37}$; C, $\$21\frac{23}{37}$; D, $\$4\frac{2}{37}$.

Page 108.

49. 90.
50. 630.
51. 150.
52. 24, 76.
53. 5, 6.
54. B, 8 yrs; A, 12 yrs.
55. 15, 35.
56. 60.
57. 12, 4.

Page 110.

3. $\frac{xz+y^2}{yz}$.
4. $\frac{2d^2+3a}{cd}$.
5. $\frac{3x^2+2y}{6xz}$.
6. $\frac{2x^2+3y}{6ax}$.
7. $\frac{b+2ax}{3aby}$.
8. $\frac{4ax+5a}{3x^2y}$.
9. $\frac{a^2+xz}{a^2-az+ax-xz}$.
10. $\frac{2+2x^2}{1-x^2}$.
11. $\frac{2+2x^4}{1-x^4}$.
12. $\frac{1+x^2}{1-x^2}$.
13. $\frac{x+xy-y^2}{x^2-y^2}$.
14. $\frac{a+1}{a}$.
15. $\frac{3x^2}{x^2-1}$.

Page 111.

16. $3a-b+\frac{2a}{a^3-a^2b-ab^2+b^3}$.
17. $\frac{2a^2+ab}{a^2-b^2}$.
18. $\frac{y^2-2xy}{x^2-xy}$.
19. $\frac{x^3+x^2-1}{x^4-x^2}$.
20. $\frac{2}{1+x^2+x^4}$.

Page 113.

3. $\frac{7a}{30}$.
4. $\frac{11x}{35}$.
5. $\frac{5a}{28b}$.
6. $\frac{x}{18a}$.
7. $\frac{d}{2ax}$.
8. $\frac{13c}{2ad}$.
9. $\frac{a(15d-4b)}{6xy}$.
10. $\frac{mn(2y-3)}{4y^2}$.
11. $\frac{a-5b}{6}$.
12. $\frac{5b-a}{a^2-b^2}$.
13. $\frac{x-9}{x^2-1}$.
14. $\frac{-4x}{x^2-1}$.

Page 114.

15. $\frac{6x}{x^2-9}$.
16. $\frac{6a^2b+2b^3}{a^2-b^2}$.
17. $3x+\frac{27a-3b}{30a}$.
18. $4x+\frac{3x-3z}{y}$.
19. $\frac{7y^2-3xy+4x^2y-3xy^2}{x^2y^2}$.
20. $\frac{b^3-8a^2b^2-4ab-7a^2}{a^3b^3}$.
21. $\frac{2x}{1-x^4}$.

22. $\dfrac{x^2+2ax-8a^2}{x^3-3ax^2+3a^2x-a^3}$.

23. $\dfrac{2x-3}{4x^3-x}$. 24. 1.

Page 116.

3. x.
4. $\dfrac{x^3}{yz}$.
5. xy.
6. $\dfrac{a^2b^2}{d}$.
7. $\dfrac{m^2n^3}{b}$.
8. cr^2s.
9. $adxy$.
10. $\dfrac{6a^2x^2}{x+y}$.
11. $\dfrac{8aby^2}{3(a+x)}$.
12. $\dfrac{5a^2x^2}{2(c+d)}$
13. $\dfrac{3a+b}{3}$.
14. $\dfrac{az}{2}$.
15. $12c^2d^2(a+b)$.
16. $\dfrac{m^2+2mn+n^2}{m^2+n^2}$.

Page 117.

17. $\dfrac{9rs^2}{2}$. 18. $\dfrac{3(x^2+2y)}{x-y}$.

Page 118.

2. $\dfrac{an}{bm}$.
3. $\dfrac{3cx}{2by}$.
4. $\dfrac{15x^3}{2a^2}$.
5. $\dfrac{a^4b^4}{2y^n}$.
6. $\dfrac{axy-ay^2}{2x}$.
7. $\dfrac{ax}{30}$.
8. $\dfrac{2a+3b}{4b}$.
9. $x-a$.

Page 119.

10. $\dfrac{ab}{x^2-y^2}$. 11. x.
12. 1.
13. $\dfrac{x^3-xy^2}{a}$.
14. $\dfrac{cd}{x^4-y^4}$.
15. $\dfrac{9a}{2}$.
16. $\dfrac{3a}{x}$.
17. $\dfrac{9x^2}{16(x+y)}$.
18. $\dfrac{3x}{c}$.
19. $\dfrac{1}{(x+y)^2}$.
20. $1\frac{1}{2}$.
21. $\dfrac{x-6}{x-3}$.
22. $\dfrac{x^2-7x+6}{x^2}$.
23. $\dfrac{x^2-2x-8}{x^2-2x}$.
24. $\dfrac{x^2y^2}{x^2-y^2}$.
25. $\dfrac{72c^2-2x^2}{9c^2}$.

Page 120.

26. $\dfrac{-a^2-x^2}{ax}$.
27. $\dfrac{a^2+b^2}{a}$.
28. $\dfrac{ax}{a^2-x^2}$.
29. $\dfrac{a}{a-b}$.
30. 1.
31. $\dfrac{1-ax^2-y+ax^2y}{x+x^2}$.

Page 122.

2. $\dfrac{8b}{11}$.
3. $\dfrac{5y}{4ab}$.
4. $\dfrac{2y^2}{81az}$.
5. $\dfrac{5z}{17abc}$.
6. $\dfrac{4b^2}{17x^2y}$.
7. $\dfrac{2xy^2}{15a^2z^3}$.
8. $\dfrac{a}{3c^2dx}$.
9. $\dfrac{x+z}{v}$.
10. $\dfrac{1}{5a^2b^2z}$.
11. $\dfrac{a-c}{1+x}$.

12. $\dfrac{a^2+4x+4}{ad+ac+dz+cz}$

13. $\dfrac{5(x+y)}{x-y}$. 14. $\dfrac{ab+cd}{a^2-c^2}$.

15. $\dfrac{3an+cm}{x^4-y^4}$.

16. $\dfrac{4ax+4by}{ac^2+ad^2+bc^2+bd^2}$

17. $\dfrac{xy+xz}{ax+nx+az+nz}$.

Page 124.

2. $\dfrac{ay}{bx}$. 11. $\dfrac{a+x}{2}$.

3. $\dfrac{3ax}{bcd}$. 12. $\dfrac{2a+x}{a^2+ax+x^2}$.

4. $\dfrac{9x}{8a}$. 13. $\dfrac{2m^2-2n^2}{3m+n}$.

5. $\dfrac{8a^3}{3dxy}$. 14. $\dfrac{2c+2d}{3}$.

6. $\dfrac{7ax}{2y}$. 15. $\dfrac{2a}{a-b}$.

7. $\dfrac{2cx}{3az}$. 16. $\dfrac{1}{2(x^2-xy+y^2}$.

8. $\dfrac{4}{3ayz}$. 17. $2a^3b+2ab^3$.

9. $\dfrac{dz}{8cx}$. 18. $\dfrac{ady+cy}{dx}$.

10. $\dfrac{abc}{m^2n}$. 19. $\dfrac{c+dy}{2xy^2}$.

Page 125.

20. $\dfrac{1}{x-2}$.

21. $\dfrac{ax}{b(x-6)}$.

22. $\dfrac{2(x-6)}{3x^2}$.

23. $\dfrac{3(x-6)}{4y}$. 25. a.

24. $\dfrac{24x^2-1}{8x+1}$. 26. $\dfrac{x^3y+y^2}{xy^2+x^2}$.

27. $\dfrac{a-x}{a^3+3a^2x+3ax^2+x^3}$.

28. $\dfrac{b+1}{ab^2}$.

29. $\dfrac{1}{5(a-b)}$. 30. $\dfrac{y-1}{ay}$.

31. $\dfrac{ax^4+ax^2y-x^5y-x^3y^2}{abc}$.

Page 126.

33. $\dfrac{cdx+ad}{cdx+bc}$. 35. $9(a-y)$.

34. $\dfrac{15a^2+5x}{60+3x}$. 36. $\dfrac{4x^2y-4xy^2}{5abx-3cby}$

37. $\dfrac{2x}{x-y}$.

38. $4a^2-8ax+4x^2$.

39. $\dfrac{6acx+4d}{6acx+9d}$. 40. $\dfrac{2x^2-y^2}{x-3y}$.

41. $\dfrac{acx^2y-3x^2}{a^2c^2+2ac^2x}$

1. $\dfrac{x-3}{x+1}$. 2. $\dfrac{m-1}{m+1}$.

3. $\dfrac{x^3-2x^2-2x+1}{4x^3-7x-1}$.

4. $\dfrac{a^2-5a+6}{3a^2-8a}$.

Page 127.

5. $\frac{3x^2}{x^2-1}$.

10. $\frac{4x^4}{x^2+2xy+y^2}$.

6. $\frac{1}{x+2}$.

11. $a^2+1+\frac{1}{a^2}$.

7. $\frac{1}{(x-y)(y-z)}$.

12. $\frac{4a}{ax+x}$.

8. $\frac{1}{abc}$.

13. x.

9. $\frac{4xy}{x^2-y^2}$.

14. $\frac{x}{a}$.

Page 129.

1. 8.
3. $\frac{1}{2}$.
5. $\frac{3}{13}$.
2. $2\frac{4}{5}$.
4. 6.
6. 12.

7. $22\frac{5}{13}$.

9. $\frac{1+6a}{4a+2b}$.

8. $\frac{b}{a-1}$.

10. $\frac{ac+abd}{d+ad}$.

Page 130.

11. 5.

13. $\frac{-3a}{4}$.

12. $\frac{a^2b-b}{2}$.

14. $1\frac{1}{23}$.

15. $\frac{5a+135b}{24}$.

16. $\frac{a^2(c-a)(c+1)}{c^2}$.

17. 12.
20. 9.
23. $\frac{1}{2}$.
18. 4.
21. -7.
24. 9.
19. 2.
22. 4.

Page 131.

25. 8.

26. $-2\frac{1}{43}$.

27. $\frac{ab(2c-a-b)}{c(a+b)-(a^2+b^2)}$.

28. $-7\frac{6}{7}$.

29. $\frac{a^2}{b-a}$.

31. 66.
32. 8.
33. 6.

Page 132.

34. $4\frac{5}{7}$.
38. 13.
42. 1.
35. 3.
39. 7.
43. 9.
36. 65.
40. -10.
44. 10.
37. 22.
41. 5.
45. 6.

46. $\frac{2a^3}{b-1}$.

47. $\frac{ac}{b}$.

Page 133.

48. 1.
49. $\frac{c+ad+bd+2a-2b}{2b}$.
50. $a^2+2ab+b^2$.
51. 50.
52. 12, greater; 4, less.
53. $2400.
54. 47, greater; 23, less.
55. 75.

Page 134.

57. $4\frac{14}{19}$.
59. $4\frac{4}{17}$.
58. $2\frac{29}{143}$.
60. 12.
61. 72.

Page 135.

62. 2280.
63. $1500.
64. 64, A's; 80, B's; 144, C's; 104, D's.
65. 150, corn; 100, oats; 50, rye.
66. $100\frac{32}{33}$, less; $103\frac{1}{33}$, greater.

Page 136.

67. $84.
68. $5000.
70. 1024.

Page 137.

71. 760. 72. 90.

74. 42, 30.
75. 12, A's age; 32, B's age.
76. \$8000, E's; \$7000, G's.
77. \$450, A's; \$270, B's.

Page 138.

78. \$60. 81. \$1400.
79. 51. 82. \$360.
80. 60. 83. 15, 21.

84. 18, 1st; 22, 2d; 10, 3d; 40, 4th.

Page 139.

85. \$4000. 87. 6.
86. 10 yds. 88. 720.

89. \$24, A; \$36, B; \$80, C; \$175, D.
90. 100, 150.
91. 7 cts., 12 cts.
92. 12, A; 24, B; 18, C.

Page 140.

93. \$120, better horse; \$90, poorer horse.
94. 13, 40. 96. $27\frac{8}{11}$.

Page 141.

97. $43\frac{7}{11}$ min. 99. 6 o'clock.
98. $54\frac{6}{11}$ min. 100. $16\frac{4}{11}$ min.
101. 30 min., or 9 o'clock.

Page 142.

103. \$20, saddle; \$180, horse.

104. $\frac{b}{1+a}=\$15$ B

$\frac{ab}{1+a}=\$60$ A.

105. $\frac{b}{1+a}=3$, 1st.

$\frac{ab}{1+a}=21$, 2d.

106. $\frac{mn}{m+n}$; $2\frac{1}{12}$; $4\frac{4}{5}$.

107. $\frac{ad+bd}{b}=3$.

108. $\frac{bc-b^2}{1+b}=64$.

Page 146.

2. $x=3$; $y=2$.
3. $x=1$; $y=1$.
4. $x=2$; $y=1$.
5. $x=2$; $y=1$.
6. $x=2$; $y=2$.
7. $x=1\frac{3}{7}$; $y=5\frac{2}{7}$.
8. $x=5$; $y=6$.
9. $x=5$; $y=4$.
10. $x=2\frac{13}{17}$; $y=4\frac{7}{17}$.
11. $x=5$; $y=4$.
12. $x=4$; $y=3$.
13. $x=6$; $y=4$.
14. $x=12$; $y=10$.

Page 147.

15. $x=8$; $y=9$.
16. $x=5$; $y=3$.
17. $x=5$; $y=4$.
18. $x=5\frac{20}{77}$; $y=15\frac{16}{77}$.

Page 148.

2. $x=2$; $y=3$.
3. $x=4$; $y=5$.
4. $x=2\frac{3}{5}$; $y=7\frac{4}{5}$.
5. $x=8$; $y=10$.
6. $x=2$; $y=3$.
7. $x=1$; $y=2$.

8. $x=5$; $y=3$.
9. $x=4$; $y=5$.

Page 149.

10. $x=5$; $y=9$.
11. $x=7$; $y=3$.
12. $x=4$; $y=10$.
13. $x=3$; $y=2$.
14. $x=6$; $y=3$.
15. $x=6$; $y=6$.
16. $x=15$; $y=12$.
17. $x=7$; $y=5$.
18. $x=6\frac{69}{236}$; $y=-3\frac{53}{59}$.
19. $x=15\frac{2}{5}$; $y=18\frac{18}{25}$.

Page 150.

2. $x=4$; $y=3$.
3. $x=3$; $y=4$.

Page 151.

4. $x=16$; $y=5$.
5. $x=1$; $y=3$.
6. $x=\frac{1}{3}$; $y=\frac{1}{5}$.
7. $x=4$; $y=3$.
8. $x=9$; $y=6$.
9. $x=2$; $y=5$.
10. $x=60$; $y=36$.
11. $x=11$; $y=6$.
12. $x=6$; $y=12$.
13. $x=12\frac{3}{7}$; $z=15\frac{3}{7}$.
14. $x=10$; $y=5$.
15. $x=42$; $y=35$.
16. $x=\frac{1}{2}$; $y=\frac{1}{4}$.
17. $x=\frac{61}{92}$; $y=\frac{61}{103}$.
18. $x=\frac{1}{3}$; $y=\frac{1}{5}$.
19. $x=\dfrac{2a}{m+n}$; $y=\dfrac{2b}{m-n}$.

Page 152.

20. $x=40$; $y=60$.
21. $x=6$; $y=12$.
22. $x=4\frac{123}{236}$; $y=4\frac{42}{59}$.
23. $x=5\frac{1}{2}$; $y=4\frac{1}{2}$.
24. $x=5$; $y=7$.
25. $x=2$; $y=3$.
26. $x=a$; $y=b$.

Page 153.

27. $x=\dfrac{ac}{a+b}$; $y=\dfrac{bc}{a+b}$.
28. $x=\dfrac{d-6a^2+2b^2}{3a}$;

$$y=\frac{3a^2+d-b^2}{3b}.$$

29. $x=\dfrac{3m}{2}$; $y=\dfrac{n}{2}$.
30. $x=\dfrac{1}{ab}$; $y=\dfrac{1}{cd}$.

Page 154.

4. $x=16$; $y=8$.

Page 155.

5. $x=17$; $y=12$.
6. $x=41$; $y=7$.
7. \$4, men; \$2, boys.
8. $\frac{4}{21}$.
9. \$33, \$56.
10. \$180, \$120.
11. \$250, A; \$320, B.
12. $\frac{5}{18}$. 13. 24. 14. 1, 5.

Page 156.

15. 12 persons; \$5, each paid.
16. $x=£3$; $y=£2$.
17. 53.
18. \$4800, A's; \$5000, B's.
19. 55 at \$20, 45 at \$30.
20. 50, father's; 30, son's.
21. 60 cts., A; 40 cts., B.

22. 65 at $45, 35 at $37.
23. 72 apples, 60 pears.

Page 157.

24. 30 miles per hour; 90 miles.

Page 159.

2. $x=3$; $y=4$; $z=5$.
3. $x=5$; $y=6$; $z=8$.
4. $x=1$; $y=2$; $z=3$.
5. $x=4\frac{1}{6}$; $y=4\frac{1}{6}$; $z=4\frac{5}{6}$.
6. $x=20$; $y=10$; $z=5$.
7. $x=2$; $y=10$; $z=14$.
8. $x=6$; $y=4$; $z=2$.
9. $x=40$; $y=30$; $z=24$; $u=26$.

Page 160.

10. $x=2$; $y=3$; $z=4$; $w=5$.
11. $x=4$; $y=5$; $z=6$; $u=2$; $v=3$.
12. $x=\frac{1}{2}$; $y=\frac{1}{3}$; $z=\frac{1}{4}$.
13. $x=3$; $y=6$; $z=9$.

Page 161.

14. $x=\dfrac{a+b-c}{2}$; $y=\dfrac{a-b+c}{2}$; $z=\dfrac{b+c-a}{2}$.
15. $x=4$; $y=5$; $z=6$; $u=8$; $w=7$.
16. $x=a$; $y=b$; $z=c$.
17. $x=3$; $y=5$; $z=6$; $u=7$; $t=8$.
18. $x=\frac{13}{35}$; $y=\frac{13}{30}$; $z=\frac{13}{56}$.

1. 10, 30. 20.
2. 50. 58, 80.
3. $300, A; $420, B; $780, C.

Page 162.

4. $.10, sugar; $.25, coffee; $.75, tea.
5. 16, 1st; 24, 2d; 5, 3d; 80, 4th.
6. $14\frac{34}{49}$, A; $17\frac{28}{41}$, B; $23\frac{7}{31}$, C.
7. 361.
8. $\frac{4}{15}$, smaller; $\frac{9}{15}$, larger.
9. $40, eldest; $30, second; $24, third; $26, fourth.

Page 163.

10. 40 sheep; 8 cows; 6 horses.
11. $x=98\frac{18}{19}$; $y=295\frac{15}{19}$; $z=352\frac{12}{19}$.

Page 165.

3. ∞.
4. Indeterminate, $\frac{0}{0}$.

Page 166.

2. $\dfrac{ab}{a+b}$; $5\frac{5}{11}$.
3. $\dfrac{b(n-1)}{a-n}$, B's age; $4\frac{1}{2}$ yrs.; $\dfrac{ab(n-1)}{a-n}$, A's age; 27 yrs.
4. $\dfrac{an}{b-a}$ days.

Page 168.

2. $36x^4y^2$.
3. $-64a^6b^6$.
4. $-27a^3b^9$.
5. $9c^4d^4$.
6. $32a^5x^{15}y^{10}$.
7. $64x^{12}y^6z^{18}$.
8. $256a^4b^{16}d^{12}$.
9. $-a^{35}b^{14}c^{42}$.
10. $256a^8b^8c^{16}$.
11. $32x^{10}y^5z^{20}$.
12. $-125a^9b^3c^6$
13. $a^{32}b^{-16}c^{-24}$.
14. $256a^8b^{12}c^{-16}$.
15. $512a^{-18}b^{18}c^3$.

Page 169.

16. $16x^8y^{12}z^4$.
17. $27x^{-6}y^{-12}$.
18. $-64a^9z^6y^3$.
19. $-32a^{-10}y^{-15}$
20. $16a^4x^8y^{-12}$.
21. $x^{-12}y^{-6}z^{3n}$.
22. $x^{5m}y^{-5n}z^{-5m}$.
23. $a^{-8n}z^{4n}w^{4m}$.
24. $a^{-8}y^{-10n}z^{-6n}$.
25. $\pm x^{4n}y^{-3n}z^{-n^3}$

26. $\pm a^{20n}b^{10n}c^{-5n^2}d^{-10n^2}$.

27. $a^{3n-6}b^{8-4n}c^{3n-6}d^{4-n^2}$.

29. $\dfrac{4a^2}{9b^2}$.

30. $\dfrac{4x^4}{9y^2}$.

31. $-\dfrac{216x^3y^3}{125a^3b^3}$.

32. $\dfrac{4096a^8b^4}{2401x^8y^4}$.

33. $\dfrac{a^{-12}b^{6n}}{c^{-24}d^{18}}$.

34. $\dfrac{x^{-7n}y^{-14n}}{a^{7n}z^{28}}$.

35. $\dfrac{a^{2n}b^{2n}c^{n^2-n}}{x^{n^2-2n}y^{n^2-3n}}$.

36. $\dfrac{a^{4n}b^{6n}c^{-8n}}{x^{-8n}y^{-8n}z^{8n}}$.

Page 170.

2. $a^2 + 2ab + b^2$.
3. $a^2 + 2a + 1$.
4. $x^3 + 3x^2y + 3xy^2 + y^3$.
5. $8 + 12x + 6x^2 + x^3$.
6. $27 + 27y + 9y^2 + y^3$.
7. $4 + 4b^2 + b^4$.
8. $a^4 - 2a^2b + b^2$.
9. $a^2 + 2ab + 2ac + b^2 + 2bc + c^2$.
10. $x^4 + 8x^3y + 24x^2y^2 + 32xy^3 + 16y^4$.
11. $n^4 - 4n^3m + 6n^2m^2 - 4nm^3 + m^4$.
12. $a^3 + 3a^2b + 3ab^2 + b^3$.
13. $x^{2n} + 2x^ny^n + y^{2n}$.
14. $8a^3 + 36a^2b + 54ab^2 + 27b^3$.
15. $27x^3 - 54x^2y + 36xy^2 - 8y^3$.
16. $n^4 + 2m^2n^2 + m^4$.
17. $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$.
18. $x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$.
19. $16a^4 + 64a^3b + 96a^2b^2 + 64ab^3 + 16b^4$.
20. $a^7 + 7a^6x + 21a^5x^2 + 35a^4x^3 + 35a^3x^4 + 21a^2x^5 + 7ax^6 + x^7$.
21. $243x^5 + 810x^4z + 1080x^3z^2 + 720x^2z^3 + 240xz^4 + 32z^5$.
22. $16y^4 + 32y^3z^2 + 24y^2z^4 + 8yz^6 + z^8$.
23. $a^2 + 2ab + 2ac + 2ad + b^2 + 2bc + 2bd + c^2 + 2cd + d^2$.

Page 171.

2. $a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$.
3. $x^2 + y^2 + z^2 - 2xy + 2xz - 2yz$.
4. $a^2 + c^2 + d^2 + e^2 + 2ac + 2ad + 2ae + 2cd + 2ce + 2de$.
5. $1 - 2a + 3a^2 - 4a^3 + 3a^4 - 2a^5 + a^6$.
6. $a^2 + 4b^2 + 9c^2 + d^2 + 4ab + 6ac + 2ad + 12bc + 4bd + 6cd$.
7. $1 + 4a - 2a^2 - 10a^3 + 13a^4 - 6a^5 + a^6$.
8. $1 + 4x^2 + y^4 + x^2y^2 - 4x - 2y^2 + 2xy + 4xy^2 - 4x^2y - 2xy^3$.
9. $4a^2 + b^2 + c^2 + d^2 - 4ab + 4ac - 4ad - 2bc + 2bd - 2cd$.

Page 175.

2. $x^3 + 3x^2y + 3xy^2 + y^3$.
3. $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.
4. $a^5 - 5a^4c + 10a^3c^2 - 10a^2c^3 + 5ac^4 - c^5$.
5. $a^4 - 4a^3x + 6a^2x^2 - 4ax^3 + x^4$.
6. $a^6 + 6a^5x + 15a^4x^2 + 20a^3x^3 + 15a^2x^4 + 6ax^5 + x^6$.
7. $a^9 - 9a^8c + 36a^7c^2 - 84a^6c^3 + 126a^5c^4 - 126a^4c^5 + 84a^3c^6 - 36a^2c^7 + 9ac^8 - c^9$.
8. $x^7 - 7x^6y + 21x^5y^2 - 35x^4y^3 + 35x^3y^4 - 21x^2y^5 + 7xy^6 - y^7$.
9. $x^{10} + 10x^9y + 45x^8y^2 + 120x^7y^3 + 210x^6y^4 + 252x^5y^5 + 210x^4y^6 + 120x^3y^7 + 45x^2y^8 + 10xy^9 + y^{10}$.

10. $x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$.
11. $x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$.
12. $1 + 5a + 10a^2 + 10a^3 + 5a^4 + a^5$.
13. $1 - 7a + 21a^2 - 35a^3 + 35a^4 - 21a^5 + 7a^6 - a^7$.
14. $x^4 + 4x^3ac + 6x^2a^2c^2 + 4xa^3c^3 + a^4c^4$.
15. $x^5 + 5x^4bc + 10x^3b^2c^2 + 10x^2b^3c^3 + 5xb^4c^4 + b^5c^5$.
17. $16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4$.
18. $27a^3 + 54a^2c + 36ac^2 + 8c^3$.
19. $\frac{a^4}{16} + \frac{a^3b}{6} + \frac{a^2b^2}{6} + \frac{2ab^3}{27} + \frac{b^4}{81}$.
20. $16 + \frac{32x}{3} + \frac{8x^2}{3} + \frac{8x^3}{27} + \frac{x^4}{81}$.
21. $x^3 - \frac{3ax^2}{y} + \frac{3a^2x}{y^2} - \frac{a^3}{y^3}$.
22. $16a^4x^4 + 96a^3bx^3y + 216a^2b^2x^2y^2 + 216ab^3xy^3 + 81b^4y^4$.
23. $8a^6x^3 + 36a^4b^2x^2y^2 + 54a^2b^4xy^4 + 27b^6y^6$.

Page 179.

2. $\pm 4xyz$. $\pm 6ab^2c$.
3. $2xy^2z^2$. $3bz^2y$.
4. $\pm 4x^2yz$. $\pm 3a^2x^2z$.
5. ± 12. ± 16. ± 18.
6. 4. 8. 16.
7. ± 6. 12.
8. $\pm (a + b)$. $\pm (a + 1)$.
9. $\pm (a^2 + ab)$. $\pm (mn + m^2)$.

Page 180.

2. $4ab^2c^2$.
3. $-2ab^2c$.
4. $2a^2c^2x$.
5. $3xy^2z$.
6. $2ab^2c^2$.
7. $-2a^{\frac{1}{3}}bc^{\frac{2}{3}}$.
8. $-ac^2x^{\frac{2}{5}}y^{\frac{1}{5}}$.
9. ax^3y.
10. $axy^{\frac{2}{3}}z^{\frac{5}{3}}$.
11. $xy^2z^{\frac{3}{4}}w^{\frac{1}{2}}$.
12. $2x^ny^{\frac{n}{2}}z^{\frac{1}{4}}$.
13. $a^2xy^{\frac{3}{2}}z^2$.
14. $a^2x^4y^3$.
15. $\frac{4a}{5y^2}$.
16. $\frac{2x}{3y^2}$.
17. $\frac{5x}{6a^2}$.

Page 183.

3. $x + 2$.
4. $b + x$.
5. $2x + 1$.
6. $a + \frac{1}{2}b$.
7. $3a - 2b$.
8. $a + b - c$.
9. $2x^2 - 3x + 1$.
10. $2a^2 + a - 2$.
11. $x^3 - 2x^2 + 3x$.
12. $4a^2 - 3ax + 5x^2$.
13. $a - b - c$.
14. $3x^2 - 2x + 6$.
15. $2x^3 + 3x^2 - x - 1$.
16. $7x^2 - 2x - \frac{3}{2}$.

Page 186.

3. 53.
4. 63.
5. 66.
6. 96.
7. 266.
8. 344.
9. 821.
10. 886.
11. 969.
12. 2424.
13. 3546.
14. 5555.
15. 472.
16. 3375.
17. .874.
18. .5555.
19. .306.
20. .315.
21. .8411 +.

Page 187.

22. $\frac{235}{886}$.
23. $\frac{884}{989}$.
24. .70710 +.
25. .86602 +.
26. .94868 +.

Page 189.

3. $x+2y$. 7. $a+\frac{1}{a}$.

4. $3a+1$. 8. $a-4$.

5. $2x-3$. 9. $2a+1$.

6. $3x+4$. 10. $3x^2-2x+1$.

11. $2m^2-3m+1$.

12. $1-a+a^2$.

13. $m-1-\frac{1}{m}$.

14. $x-y+2z$.

Page 190.

16. $1-2a$. 17. $x+1$. 18. $x+1$.

Page 194.

3. 42. 12. 1.259 +.

4. 64. 13. 2.0800 +.

5. 55. 14. .6463 +.

6. 89. .8617 +.

7. 57. 15. .8735 +.

8. 63. 16. .843 +.

9. 177. 17. 19.51+.

10. 126. 18. 12.

11. 536.

Page 197.

2. $3x\sqrt{2}$. 7. $9a\sqrt{xy}$.

3. $6a\sqrt{b}$. 8. $6yz\sqrt{x}$.

4. $5xy^2\sqrt{3x}$. 9. $15xyz\sqrt{2x}$.

5. $10a^2b\sqrt{b}$. 10. $a\sqrt{1-x}$.

6. $12dx\sqrt{3ay}$. 11. $x\sqrt[3]{x-y^2}$.

Page 198.

12. $a^2\sqrt[3]{a-x^2}$. 13. $a(a+b)\sqrt{a}$.

14. $a(a+b)\sqrt{a^2-b^2}$.

15. $(x^2-y^2)\sqrt{x}$.

16. $x(1+y+y^2)^{\frac{1}{3}}$.

17. $4a(1+y)x^{\frac{1}{3}}$.

19. $\frac{1}{3}\sqrt{6}$. 23. $\frac{1}{6ab}\sqrt{30abxy}$

20. $\frac{1}{7}\sqrt{21}$. 24. $\frac{1}{2b}\sqrt[3]{7ab^2}$.

21. $\frac{1}{3}\sqrt[3]{15}$. 25. $\frac{x}{5a}\sqrt{15ax}$.

22. $\frac{1}{7b}\sqrt{35ab}$. 26. $\frac{1}{2}\sqrt{10}$.

Page 199.

27. $\frac{x^2}{y}\sqrt{3}$.

28. $\frac{a+b}{a-b}\sqrt{a(a-b)}$.

29. $\sqrt[3]{x(x+y)}$.

30. $\frac{a^2-b^2}{a^2+b^2}\sqrt{x(a^2+b^2)}$.

31. $\frac{3a}{7y}(21xy)^{\frac{1}{2}}$.

32. $\frac{5x}{2}(6y^2)^{\frac{1}{3}}$.

Page 200.

2. $\sqrt{9a^2x^4}$.

3. $\sqrt{16x^4y^2}$.

4. $\sqrt[3]{8a^6x^6y^3}$.

5. $\sqrt{a^2+2ab+b^2}$.

6. $\sqrt[3]{a^3-3a^2b+3ab^2-b^3}$.

7. $\sqrt{4a^3b}$.

8. $\sqrt{9b^3xy}$.

9. $\sqrt{4x^4+4x^2y^2}$.

10. $\sqrt{2x^3+4x^2y+2xy^2}$.

11. $\sqrt{9a^4-9a^2b^2}$.

12. $\sqrt{a^3-a^2b-ab^2+b^3}$.

13. $\sqrt{a(x-y)}$.

14. $\sqrt{x(x+y)^2}$.

Page 201.

2. $\sqrt[6]{1000}, \sqrt[6]{9}$.
3. $\sqrt[6]{a^6x^3}$, $\sqrt[6]{x^4y^2}$.
4. $\sqrt[4]{x^4y^6}$, $\sqrt[4]{xy^2}$.
5. $\sqrt[12]{a^6x^8y^3}$, $\sqrt[12]{x^4y^8z^4}$.
6. $\sqrt[10]{a^5x^5y^{20}}$, $\sqrt[10]{a^4x^2y^4}$.
7. $\sqrt[mn]{a^nx^m}$, $\sqrt[mn]{b^ny^m}$.
8. $\sqrt[6]{a^3x^3}$, $\sqrt[6]{b^2y^2}$, $\sqrt[6]{a^3z^3}$.
9. $\sqrt[6]{a^6x^3}$, $\sqrt[6]{a^2y^4}$, $\sqrt[6]{b^3y^9}$
10. $\sqrt[6]{\frac{8}{27}}$, $\sqrt[6]{\frac{9}{25}}$, $\sqrt[6]{8}$.
11. $\sqrt[6]{\frac{x^3}{27}}$, $\sqrt[6]{\frac{x^2}{16}}$, $\sqrt[6]{x^9}$.
12. $\sqrt[6]{(a+b)^3}$, $\sqrt[6]{(a+b)^2}$.

Page 202.

2. $15\sqrt{2}$. 4. $9\sqrt[3]{4}$. 6. $12\sqrt[3]{6}$.
3. $11\sqrt{3}$. 5. $56\sqrt{2}$. 7. $3\frac{1}{10}\sqrt{6}$.

Page 203.

8. $12\sqrt{6}$.
9. $1\frac{1}{3}\sqrt{3}$.
10. $1\frac{1}{4}\sqrt{2}$.
11. $44x^2y^2\sqrt{2xy}$.
12. $3\sqrt{3}$.
13. $xy\sqrt{2}$.
14. $-\sqrt[3]{5}$.
15. $\sqrt{6}$.
16. $11\sqrt{3}$.
17. $34\sqrt{2}$.
18. $9a\sqrt{5a}$.
19. $(2a+2b-5c)\sqrt{2y}$.
20. $2y\sqrt{2}$.
21. $6xy^2\sqrt[3]{2}$.
22. $\frac{37}{18b}\sqrt{3a}$.

Page 205.

3. $12\sqrt{10}$.
4. $36\sqrt{2}$.
5. $18x\sqrt{3}$.
6. $24\sqrt[3]{y}$.
7. $24\sqrt[3]{2ac}$.
8. $30xy\sqrt{2y}$.
9. $40xy\sqrt[3]{x}$.
10. $120xy\sqrt[3]{2y}$.
11. $12\sqrt{70}$.
12. $30\sqrt{21}$.
13. 200.
14. $3ab$.
15. $x^2y\sqrt[6]{108xy^4}$
16. $ay\sqrt[4]{a^2x^3y}$.
17. $y\sqrt[6]{27a^5x^5y}$.
18. $12y\sqrt[6]{8x^5y}$.
19. $\frac{2x}{35}\sqrt{105}$.

Page 206.

21. 1.
22. 46.
23. $27-10\sqrt{2}$.
24. $4x^2-y$.
25. $\sqrt{6}-3\sqrt{5}-\sqrt{10}+5\sqrt{3}$.
26. $x-y$.
27. $x-2\sqrt{xy}+y$.
28. x^2+xy+y^2.
29. $-b$.
30. $a^{\frac{4}{3}}+2a^{\frac{2}{3}}b^{\frac{2}{3}}+b^{\frac{4}{3}}$.
31. $a-b$.
32. $a-a^{\frac{1}{4}}b^{\frac{3}{4}}-a^{\frac{3}{4}}b^{\frac{1}{4}}+b$.
33. a^4+6a^2+16.
34. $12+24\sqrt{5}+30\sqrt{2}+60\sqrt{10}$.

Page 207.

3 $6\sqrt{3}$.
4. $\frac{8}{3}$.
5. $2\sqrt{2a}$.
6. $2x\sqrt{3}$.
7. $\sqrt{6}$.
8. $8\sqrt[3]{4}$.
9. $\frac{3}{4}\sqrt{10}$.
10. $\frac{3}{xy}\sqrt[6]{a^2x^5y^3}$.

Page 208.

11. $\frac{a}{b}$.
12. $\frac{1}{x-y}\sqrt{x^2-y^2}$.
13. $\frac{3}{4}\sqrt[3]{12}$.
14. $\frac{1}{a}(2a^3x)^{\frac{1}{6}}$.
15. $\sqrt{3xy}$.
16. $2\sqrt[6]{(x-y)(x+y)^5}$.

18. $a^{\frac{1}{4}} + y^{\frac{1}{4}}$.

19. $a^{\frac{1}{4}} - y^{\frac{1}{4}}$.

20. 2.

21. $2 + 3\sqrt{2}$.

22. $4 + 3\sqrt{5}$.

23. $4 + 3\sqrt{6}$.

Page 209.

3. $9x$.

4. $4x\sqrt[3]{4x}$.

5. $16a^2b\sqrt[3]{9b}$.

6. $216x\sqrt[3]{x}$.

7. $24a^4\sqrt{3a}$.

8. $324ax$.

9. $a + b$.

Page 210.

10. $(2a + 3z)^2$.

11. $23 + 4\sqrt{15}$.

12. $52 + 16\sqrt{3}$.

13. $49 + 12\sqrt{5}$.

14. $x + 2x^{\frac{1}{2}}y^{\frac{2}{3}} + y^{\frac{4}{3}}$.

15. $a^{\frac{3}{2}} + 2a^{\frac{3}{4}}y^{\frac{2}{3}} + y^{\frac{4}{3}}$.

Page 211.

2. $4x\sqrt[4]{az}$.

3. $6ax\sqrt[4]{yz}$.

4. $3ab\sqrt[6]{xyz}$.

5. $a^2b^2\sqrt[6]{ax}$.

6. $\frac{1}{3}\sqrt{3a}$.

7. $4\sqrt[6]{2x^3y}$.

8. $\sqrt[4]{(x + y)^3}$.

Page 212.

3. $\sqrt{4ax}$.

4. $\sqrt[3]{(3a^2y)^2}$

5. $\sqrt{3x}$.

6. $\sqrt[3]{(3y^2)^2}$.

7. $\sqrt[4]{a^3b^3y^3}$.

8. $\sqrt{3yz}$.

9. $\sqrt[3]{(3a^2y)^2}$.

Page 213.

2. $\sqrt{5} + \sqrt{2}$.

3. $\sqrt{9} + \sqrt{6}$.

4. $x - \sqrt{3}$.

5. $x + 3\sqrt{6}$.

6. $\sqrt{a} + \sqrt{x}$.

7. $2\sqrt{a} - 3\sqrt{y}$.

8. $\sqrt{4a} + \sqrt{3x}$.

9. $x + \sqrt{y}$.

10. $x^2 + \sqrt{yz}$.

Page 214.

2. $\frac{3\sqrt{5}}{5}$.

3. $\frac{2\sqrt{7}}{7}$.

4. $\frac{4\sqrt{a}}{a}$.

5. $\frac{6}{\sqrt{15}}$.

6. $\frac{2a}{\sqrt{ax}}$.

7. $\frac{10}{3\sqrt{35}}$.

Page 215.

8. $\frac{2\sqrt{2} + 2\sqrt{3}}{-1}$.

9. $\frac{2x\sqrt{a} - 2x\sqrt{b}}{a - b}$.

10. $\frac{2x\sqrt{x} + 2\sqrt{xy}}{x^2 - y}$.

11. $\frac{2ab\sqrt{x} + 2ab\sqrt{y}}{x - y}$.

12. $\frac{3\sqrt{x - 1} + 3\sqrt{x + 1}}{-2}$.

Page 216.

2. $3a\sqrt{-1}$.

3. $10b\sqrt{-1}$.

4. $17ax\sqrt{-1}$.

5. $-a\sqrt{-1}$.

6. $-2mx\sqrt{-3}$

Page 217.

2. $-2\sqrt{5}$.

3. $-12\sqrt{6}$.

4. -36.

5. $-6a^2\sqrt{x}$.

6. 2.

7. $-2\sqrt{-1}$.

2. $3\sqrt{2}$.

3. [illegible].

4. $\frac{2a}{3}$.

5. $1 - \sqrt{-1}$.

6. $1 - \sqrt{-1}$.

1. $a^{m-1}x^{2m-2}y^{2m-2}$.

2. $a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b^7$.

3. $16a^4-96a^3b+216a^2b^2-216ab^3+81b^4$.

4. $\frac{a^3}{8}-\frac{a^2b}{4}+\frac{ab^2}{6}-\frac{b^3}{27}$.

Page 218.

5. $\frac{x^5}{1024}+\frac{5x^4y}{768}+\frac{5x^3y^2}{288}+\frac{5x^2y^3}{216}+\frac{5xy^4}{324}+\frac{y^5}{243}$.

6. $\frac{8x^3}{125}+\frac{36x^2y}{175}+\frac{54xy^2}{245}+\frac{27y^3}{343}$.

7. $a^n+na^{n-1}b+\frac{n(n-1)}{2}a^{n-2}b^2+\frac{n(n-1)(n-2)}{2\times3}a^{n-3}b^3+\frac{n(n-1)(n-2)(n-3)}{2\times3\times4}a^{n-4}b^4+\frac{n(n-1)(n-2)(n-3)(n-4)}{2\times3\times4\times5}a^{n-5}b^5$.

8. $a^{n-2}+(n-2)a^{n-3}b+\frac{(n-2)(n-3)}{2}a^{n-4}b^2+\frac{(n-2)(n-3)(n-4)}{2\times3}a^{n-5}b^3+\frac{(n-2)(n-3)(n-4)(n-5)}{2\times3\times4}a^{n-6}b^4$.

9. $x^r+rx^{r-1}y+\frac{r(r-1)}{2}x^{r-2}y^2+\frac{r(r-1)(r-2)}{2\times3}x^{r-3}y^3+\frac{r(r-1)(r-2)(r-3)}{2\times3\times4}x^{r-4}y^4+\frac{r(r-1)(r-2)(r-3)(r-4)}{2\times3\times4\times5}x^{r-5}y^5$

10. $1+4x^2+9y^2+z^2+4x+6y+2z+12xy+4xz+6yz$.

11. $3x^{\frac{1}{2}}-4y^{\frac{2}{3}}+2$.

12. x^2-x-1.

13. $m-\sqrt{mn}+n$.

14. $x^2-xy\sqrt{2}+y^2$.

15. $2x+2\sqrt{ax}$.

16. x.

17. $2xy^2\sqrt{xy}$.

18. $4+4a^2\sqrt{2x+y}+2a^4x+a^4y$.

19. $\sqrt[xn]{\frac{a^3}{8}}$.

20. $\frac{a+\sqrt{ab}}{a-b}$.

21. $\frac{24+17\sqrt{2}}{1}$.

22. $\frac{2a^2+2a\sqrt{a^2-x^2}-x^2}{x^2}$.

23. $\frac{2m^2-2\sqrt{m^4-1}}{2}$.

Page 221.

4. 4.	8. 27.	12. 9.	16. $5\frac{1}{4}$.
5. 32.	9. 12.	13. 25.	17. 64.
6. 56.	10. 5.	14. 121.	18. 27.
7. 6.	11. 6.	15. 25.	

Page 223.

19. $\frac{3}{4}$. 20. 2. 21. 100.

22. $a-1$.

23. $\frac{81}{a}$.

24. 4.

25. $4\frac{1}{2}$.

26. 10.

27. $\frac{2ab}{b^2+1}$.

28. $\frac{1}{3}$.

29. $\frac{a(b-1)^2}{4b}$.

30. $\frac{4}{5}$.

31. 4.

32. $\frac{4a^2}{(1+a)^2}$.

33. $\frac{4ab^2}{(1+a)^2}$.

Page 225.

3. ± 7.
4. ± 2.
5. ± 6.
6. ± 6.
7. ± 3.
8. ± 4.
9. ± 5.
10. ± 10.
11. ± 5.
12. ± 6.
13. ± 8.
14. $\pm \frac{1}{2}$.
15. $\pm 3\sqrt{2}$.
16. ± 2.
17. $\pm a\sqrt{2}$.
18. ± 2.
19. $\pm \frac{1}{2}\sqrt{3}$.

Page 226.

20. $\pm \sqrt{a^2+b^2}$.

21. $\pm \frac{1}{2}\sqrt{5}$.

22. $\pm \sqrt{\frac{a}{a-2}}$.

23. $\pm \sqrt{(a-2)^2-1}$.

1. $\pm \frac{3}{4}$.
2. ± 4.
3. ± 8.
4. ± 8.
5. ± 30; ± 50.
6. Son's, 8; father's, 32.
7. \$150.

Page 227.

8. 12 ft.; 18 ft.
9. 3 and 9.
10. 7 and 8.
11. 6 and 7.
12. 8 and 12.
13. 12 and 20.
14. 8 yd.
15. Breadth, 36 rods; length, 40 rods.
16. 8 and 10.

Page 231.

4. 5, or -9.
5. 3, or -9.
6. 2, or -10.
7. 1, or -11.
8. 1, or -21.
9. 1, or -19.
10. 1, or -25.
11. 15, or -3.
12. 11, or -3.
13. 17, or -3.
14. 30, or -2.
15. 32, or -2.
16. 2, or $-3\frac{1}{2}$.
17. 3, or $-4\frac{1}{3}$.
18. 3, or $-6\frac{1}{2}$.
19. 2, or -5.
20. 4, or $-\frac{1}{2}$.
21. 20, or 1.
22. 2, or $-5\frac{1}{3}$.
23. 4, or -1.

Page 234.

3. 1, or $-2\frac{2}{3}$.
4. 2, or $-5\frac{1}{2}$.
5. 4, or $-5\frac{1}{4}$.
6. 5, or $-4\frac{1}{3}$.
7. 10, or $-4\frac{2}{3}$.
8. 6, or $-4\frac{1}{4}$.
9. 2, or $-2\frac{4}{5}$.
10. 8, or $-7\frac{1}{4}$.
11. 6, or $-4\frac{4}{5}$.

12. $-\frac{1}{3}a \pm \frac{1}{3}\sqrt{a^2+3b}$.

13. 8, or -2.
14. 14, or -1.
15. 1, or -18.
16. 12, or -1.
17. 5, or 4.
18. 2, or $-4\frac{1}{2}$.
19. $2\frac{1}{3}$, or -3.

20. $6.229+$, or $-2.729+$.
21. $3.525+$, or $-2.325+$.
22. 3, or $-2\frac{1}{3}$.
23. 13, or $-4\frac{1}{3}$.

24. 7, or $-1\frac{2}{7}$.
25. $1\frac{7}{18}$, or $-1\frac{2}{3}$.
26. 2, or -5.
27. 14, or -10.
28. 3, or $-\frac{1}{2}$.
29. 7, or $-1\frac{2}{7}$.

Page 236.

3. ± 2, or $\pm\sqrt{-2}$.
4. 2, or $\sqrt[3]{-5}$.
5. 2, or $\sqrt[3]{-4}$.
6. ± 2, or $\pm\sqrt{-2}$.
7. $\pm\frac{1}{2}\sqrt{6}$, or $\pm\sqrt{-1}$.
8. $\pm\sqrt{2}$, or $\pm\sqrt{-6}$.

9\. 16, or 256.

10\. 4, or $\sqrt[3]{121}$.

11\. 1, or -64.

12\. 625, or 256.

13\. $\sqrt[n]{-\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2+4ac}}$.

Page 237.

15\. ± 2, or $\pm\sqrt{-7}$.

16\. ± 1, or $\pm\sqrt{-10}$.

17\. 3, or -6.

18\. ± 5, or ± 2.

19\. 4, -3, or $\frac{1}{2} \pm \frac{1}{2}\sqrt{-43}$.

Page 238.

20\. 4, or -1.

21\. 4, or 20.

22\. 4, 2, or $-\frac{7}{2} \pm \frac{1}{2}\sqrt{17}$.

23\. 3, 2, or $-3 \pm \sqrt{3}$.

24\. 3, -1, or $1 \pm \frac{1}{2}\sqrt{-10}$.

25\. 9, -2, or $\frac{7}{2} \pm \frac{1}{2}\sqrt{173}$.

26\. 3, $-4\frac{1}{2}$, or $-\frac{3}{4} \pm \frac{1}{4}\sqrt{-55}$.

27\. ± 4, or $\pm\frac{1}{2}\sqrt{15}$.

28\. 4, or 69.

Page 239.

2\. 7 and 3.

3\. 7 and 20.

4\. 28 rods and 40 rods.

5\. 20 sheep.

6\. 40 in a row; 50 rows.

7\. 10 and 12.

8\. 11 persons.

9\. 6 days.

10\. 8 ct. per doz.

11\. \$20.

12\. 20 persons.

13\. 3 inches.

Page 240.

14\. \$30.

15\. A, \$1.14039 + per rod;
B, \$.89039 + per rod.
A dug 43.84 rods;
B dug 56.16 rods.

16\. 6 rods.

17\. 12 yards and 24 yards.

18\. 9 gallons.

19\. One, 9 mi.; other, 10 mi.

20\. Silver, 2; copper, 25.

Page 242.

2\. $x^2 - 7x = -12$.

3\. $x^2 + 3x = 10$.

4\. $x^2 - 10x = -21$.

5\. $x^2 + 10x = -24$.

6\. $x^2 + x = 6$.

7\. $x^2 + 9x = -20$.

8\. $x^2 - 4x = 12$.

9\. $x^2 + 10x = -21$.

10\. $x^2 + (b-a)x = ab$.

11\. $x^2 + (c-b)x = bc$.

12\. $x^2 - 3x\sqrt{5} = -10$.

13\. $x^2 - 4x = 3$.

Page 245.

6\. $x = 6$;
$y = 2$.

7\. $x = 2$, or 3;
$y = 3$, or 2.

Page 246.

8\. $x = 5$, or $1\frac{1}{2}$;
$y = 3$, or 10.

9\. $x = 6$, or $-1\frac{1}{2}$;
$y = 1$, or -4.

10\. $x = 3$, or -5;
$y = 2$, or 6.

11\. $x = 5$, or 6;
$y = 4$, or $3\frac{1}{3}$.

12\. $x = 5$, or 10;
$y = 10$, or 5.

13\. $x = \pm 6$, or $\pm 4\sqrt{-2}$.
$y = \pm 4$, or $\pm 3\sqrt{-2}$.

14\. $x = 5$, or -2;
$y = 2$, or -5.

15. $x=4$, or 3;
$y=3$, or 4.

16. $x=4$, or -3;
$y=3$, or -4.

17. $x=3$, or 1;
$y=1$, or 3.

18. $x=3$, or 1;
$y=1$, or 3.

19. $x=3$, or -1;
$y=1$, or -3.

20. $x=\pm 6$;
$y=\pm 5$.

21. $x=\pm 2$;
$y=\pm 3$.

22. $x=\pm 7$, or $\pm 5\sqrt{2}$;
$y=\pm 3$, or $\pm 2\sqrt{2}$.

23. $x=8$, or $17\frac{2}{3}$;
$y=6$, or $-13\frac{1}{3}$.

24. $x=18$, or $12\frac{1}{2}$;
$y=3$, or $-2\frac{1}{2}$.

25. $x=2$, or -46;
$y=3$, or 15.

26. $x=4$, or 2;
$y=2$, or 4.

27. $x=6$, or 4;
$y=4$, or 6.

28. $x=\pm 2$, or $\pm \frac{1}{2}\sqrt{2}$;
$y=\pm 3$, or $\pm \frac{5}{2}\sqrt{2}$.

29. $x=64$, or 8;
$y=8$, or 64.

30. $x=4$, or -2;
$y=2$, or -4.

31. $x=2$, or 1;
$y=1$, or 2.

Page 247.

32. $x=3$, or -2;
$y=2$, or -3.

33. $x=2$, or 3;
$y=3$, or 2.

34. $x=9$, or 4;
$y=4$, or 9.

35. $x=6$, or -102;
$y=5$, or 59.

36. $x=3$, or 1;
$y=1$, or 3.

37. $x=3$, or 9;
$y=9$, or 3.

38. $x=4$, or 3;
$y=3$, or 4.

1. 2 and 6.
2. 10 and 2.
3. 4 and 9.
4. 9 and 11.
5. 36 and 64.
6. 4 and 2.

Page 248.

7. 8 and 6.

8. 8 and 6.

9. $\dfrac{a}{2} \pm \frac{1}{2}\sqrt{2b-a^2}$ and

$\dfrac{a}{2} \mp \frac{1}{2}\sqrt{2b-a^2}$.

10. 6 and 4.

11. 48.

12. A's rate, 36;
B's rate, 24.

13. Linen, 16 yards;
cotton, 48 yards.

14. 24 rods long;
18 rods wide.

15. 49 yards;
\$3 per yard.

16. Fore wheel, 12 feet;
hind wheel, 15 feet.

Page 249.

17. $\frac{1}{2}(3 \pm \sqrt{5})$ and
$\frac{1}{2}(1 \pm \sqrt{5})$.

18. A's, \$192;
B's, \$224.

19. Length, 31 rods;
breadth, 19 rods.

20. 8 men; 12 women.
Men, \$3; women, \$2.

21. Gold, 5; silver, 4.
22. 18 and 3.
23. 20 miles.

Page 262.

2. $4\frac{1}{2}$.
3. $1\frac{1}{2}$.
4. $2\frac{2}{3}$.
5. ± 6.
6. $1\frac{1}{2}$.
7. 4.
8. \$12 and \$28.
9. 12 and 18.
10. 12 and 20.

11. 200 bushels of wheat;
300 bushels of oats.

Page 264.

14. 5 and 3.
15. 5 and 2.
16. 8 and 6.
17. 8 and 4.
18. 4 and 2.
19. 4 and 3.

Page 266.

3. 6.
4. $\frac{9b^2}{a}$.
5. $\frac{4}{5}$.
6. $\frac{2ab}{b^2+1}$.
7. $\pm \frac{1}{2}\sqrt{5}$.
8. 4.
9. 4.
10. $\frac{a(\sqrt{b}-1)^2}{2\sqrt{b}}$.
11. $\frac{a(b-c)}{2\sqrt{bc}}$.
12. $\pm \frac{2a}{a+1}$.

Page 270.

2. 55.
3. 47.
4. 6.
5. $11\frac{1}{4}$.
6. 3.
7. -68.
8. $30a$.
9. $35x$.
10. 0.
11. $2n-1$.
12. \$1.72.
13. $214\frac{1}{2}$ feet.

Page 272.

2. 144.
3. 108.
4. 124.
5. $52\frac{1}{2}$.

6. $80a$.
7. $9a+9b+36c$.
8. n^2x.
9. -12.
10. 330.
11. 78.
12. \$3360.

Page 275.

2. 30.
3. 10.
4. 8.
5. 3, 5, 7.
6. 1, 3, 5.
7. 3, 6, 9.
8. 2, 5, 8.
9. 3, 5, 7.
10. 1, 2, 3, 4.
11. 3, 5, 7, 9.
12. 2, 4, 6, 8.

Page 276.

13. 2, 5, 8, 11, 14.
14. 1, 3, 5, 7, 9.
15. 1, 4, 7, 10.
16. 234.

Page 277.

2. 160.
3. 512.
4. 2187.
5. 512.

Page 278.

6. $128a^7$.
7. $768a^8x^8$.
8. 2^{n-1}.
9. $3 \times 4^{n-1}$.
10. $\frac{1}{729}$.
11. \$2187.
12. \$32000.

Page 279.

2. 2047.
3. 9841.
4. 16380.
5. 265719.

Page 280.

6. $2046a$.
7. $59048x^2$.
8. $2(2^n-1)$.
9. $3\frac{255}{256}$.
10. $10\frac{1365}{2048}$.
11. 1364.
12. 1022.
13. 4.
14. 12.
15. 3.

16. $\dfrac{x^2}{x^2-1}$.

17. $\dfrac{x^2}{x+y}$.

18. \$510.

19. \$1048575.

Page 288.

2. 7.
3. 1.
4. $\frac{2}{3}$.
5. 5.
6. 1, 3, 9.
7. 1, 2, 4.
8. 1, 3, 9, 27.
9. 1, 2, 4, 8.
10. 2, 4, 8.
11. 4, 6, 9, $13\frac{1}{2}$.
12. 2, 6, 18.
13. \$629.38 +.

Page 290.

2. 2.50243.
3. 2.45484.
4. 2.68664.
5. 2.52504.
6. 1.52634.
7. 0.42813.
8. $\bar{1}$.58433.
9. 3.68404.
10. 3.58500.
11. 3.44483.
12. 3.50093.
13. 3.27301.
14. 0.37014.
15. 0.22636.

Page 291.

2. 241.31.
3. 153.55.
4. 1.7040.
5. .19339.
6. .09652.
7. 1528.6.
8. 731.72.
9. .001765.
10. 965.06.

Page 292.

2. 8.51.
3. 87.5.
4. 756.
5. 74.87 +.
6. 418.2.
7. 5.824.
8. .000598.
9. .0000225.

Page 293.

2. .25.
3. .763.
4. 30.2.
5. 3650.
6. .15.
7. 3130.
8. 21600.
9. 41.6.
10. 4420.
11. .428.

2. 361.
3. 1225.
4. 2025.
5. 841.
6. 32767 +.
7. 15625.
8. 2744.

Page 294.

2. 14.
3. 16.
4. 64.
5. 36.
6. 16.
7. 24.
8. 42.

1. $10x^{\frac{1}{2}}y - 2$.
2. $(5-5c)x^4y^2 + 2x^{-n}y^{\frac{3}{2}} + 3x^2 + 2a + cy + 2x^3$.
3. $10cy^{-\frac{2}{3}} + 12m + 16ax - 3b$.
4. $10xy^2 - xz + 10a - (12-2b)x$.

Page 295.

5. $a^5 - b^5$.
6. $6x^m - 4xy^{n-2} - 9x^{m-1}y^2 + 6y^n$.
7. $9x^{-1} - 4y^{\frac{4}{3}}$.
8. $4x^{\frac{m}{n}} - 9y^{m-n}$.
9. $x^{4n} + 2x^{2n}y^{2m} + y^{4m}$.
10. $x^3 - x^2y + xy^2 - y^3$.
11. $x^6 - x^5y + x^4y^2 - x^3y^3 + x^2y^4 - xy^5 + y^6$.
12. $x^{n-1} + x^{n-2}y + x^{n-3}y^2 + x^{n-4}y^3 + x^{n-5}y^4 + x^{n-6}y^5$.
13. $x^2 + \frac{3}{4}$.
14. $(2x+y)(2x+y)$.
15. $(x^2+y^2)(x+y)(x-y)$.
16. $(x-7)(x+5)$.
17. $(x-9)(x+3)$.
18. $(x^4+y^4)(x^2+y^2)(x+y)(x-y)$.
19. $x-y$.
20. $2x^2+3$.
21. $x-3$.
22. x^2-5x-8.
23. $12a^2x^2y^3$.
24. $y(x^2-y^2)$.

25. $\dfrac{x-5}{2x+3}$.

26. $\dfrac{x-2}{x+4}$.

Page 296.

27. $\dfrac{a-b-c}{a+b-c}$.

28. $\dfrac{3x^2}{x^2-1}$.

29. $\dfrac{1}{x+2}$.

30. 0.

31. 0.

32. $x^4-\dfrac{1}{x^4}$.

33. $\dfrac{8a^2b^2}{a^4-b^4}$.

34. x.

35. $\dfrac{(x^2+y^2)^2}{x^4+y^4}$.

36. $\dfrac{x-y-z}{x+y+z}$.

37. $x-1$.

Page 297.

38. $\sqrt{y}-\sqrt{x}$.

39. $x^7+7x^6y+21x^5y^2+35x^4y^3+35x^3y^4+21x^2y^5+7xy^6+y^7$.

40. $32a^5+240a^4b+720a^3b^2+1080a^2b^3+810ab^4+243b^5$.

41. $a^{10}+10a^9b+45a^8b^2+120a^7b^3+210a^6b^4+252a^5b^5+210a^4b^6+120a^3b^7+45a^2b^8+10ab^9+b^{10}$.

42. $x^3+3x^2y+3xy^2+y^3$.

43. $(x+y)^2\sqrt[3]{xy}$.

44. $(x-y)\sqrt{3x}$.

45. $10\sqrt[3]{4}$.

46. $x-y$.

47. $a+2\sqrt{ab}+b$.

48. 7.

49. $\dfrac{2n}{a-b}$.

50. a^2-b^2.

51. $\dfrac{ad}{bc}$.

52. 6.

53. 12.

54. $\dfrac{bn-am}{m-n}$.

55. 5.

Page 298.

56. $\pm\ 2$, or $\pm\sqrt{-6}$.

57. $\pm\sqrt{7}$.

58. 81.

59. 25.

60. 4.

61. $\frac{2}{3}$.

62. $x=3,\ y=4,\ z=5$.

63. $x=11\frac{7}{15},\ y=-7\frac{1}{5},\ z=74\frac{2}{3}$.

64. $x=2,\ y=4,\ z=3,\ w=3,\ v=1$.

65. $x=\dfrac{a}{c},\ y=ac,\ z=\dfrac{c}{a}$

Page 299.

66. $x=\pm 2,\ y=\pm 4$.

67. $x=2$, or 3;
$y=3$, or 2.

68. $x=4$, or 5;
$y=5$, or 4.

69. $x=9$, or 25;
$y=25$, or 9.

70. $x=4$, or -2;
$y=2$, or -4.

71. $x=\pm 5,\ y=\pm 4$.

72. $x=\pm 3,\ y=\pm 4$.

73. $x=2$, or 3;
$y=3$, or 2.

74. $x=2$, or 16;
$y=2$, or $\frac{1}{2}$.

75. $x=4$, or 2;
$y=2$, or 4.

Page 300.

76. $x=5$, or 4;
$y=4$, or 5.

77. $x=4$, or 1;
$y=8$.

78. $x=8$, or 6;
    $y=6$, or 8.

79. 3, or -2;
    2, or -3.

80. 21 and 28.
81. 20 minutes past 5.
82. 20.
83. 9 and 12.
84. 9 and 3.
85. 17, 14, 27, 8, 33.
86. 2 and 3.

Page 301.

87. 276.
88. 50 apples, 150 pears.
89. 42 miles.
90. A's age, 21; B's age, 39.
91. 333.
92. 8 cents.
93. $\frac{2}{3}$.
94. 144 sq. yd.
95. 40 horses.
96. \$180 and \$120.

Page 302.

97. 8 rods.
98. 30 shillings.
99. 27 and 13.
100. 24000 men.
101. 12, 4, and 18 miles.
102. $1-\sqrt{2}$.
103. 8 persons.

Page 303.

104. $27\frac{3}{11}$ minutes past 11.
105. 40 rods and 16 rods.
106. 10 and 8.
107. \$577.18 +.
108. Length, 118.48 + feet;
     Breadth, 88.86 + feet.
109. $x=18$, or 6;
     $y=6$, or 18.
110. $x=20$, or -16;
     $y=16$, or -20.
111. 7, 13, 19, 25.
112. 2 yards and 5 yards.
113. \$40.
114. 1, 3, 9.

Page 304.

115. $x=\pm 3$, $y=\pm 1$.
116. 5 feet and 4 feet.
117. 2, 4, 8.
118. $bx+ax^2+c$.
119. 10, 20, 40.
120. \$1600, \$400, \$100.
121. 38 gallons and 62 gallons.
122. 24 bales, or 72 casks.

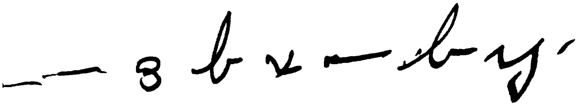
Page 305.

123. 18 acres; \$12 per acre.
124. 4.
125. 5 and 3.
126. A, 96; B, 108.
127. 15 pieces.
128. 2, 5, 8.
129. B, 15 days; C, 18 days.
120. $\frac{1}{2}(3\pm\sqrt{-3})$ and
     $\frac{1}{2}(3\mp\sqrt{-3})$.

Page 306.

131. $\pm\frac{1}{2}\sqrt{5}$ and $\frac{1}{4}(5\pm\sqrt{5})$.
132. A, 55 hours; B, 66 hours.
133. 6 days.
134. 3, or -2.
135. 1, 2, 3.
136. $x=\pm 3$, or ± 4
     $y=\pm 4$, or ± 3.
137. $x=\pm 2$, or ± 1
     $y=\pm 1$, or ± 2.
138. 300 miles.

$$\frac{3x-y}{a+b} \quad \frac{3ax-ay}{a^2+2ab+b^2}$$



MILNE'S

INDUCTIVE ALGEBRA